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Generators and Relations for certain Special Linear Groups

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Let $K = Q(\sqrt{-m})$ be a quadratic imaginary number field. Let $\mathcal{O}$ be the ring of integers of K. In this paper, I will develop an algorithm for determining a presentation of the group $SL(2, \mathcal{O})$ and will apply it to some small values of m. The principal results of this paper were announced in [16].

The method is based on a reduction theory for binary Hermitian forms due to Bianchi [2] and Humbert [10], together with a generalization of a theorem of Macbeath [13]. This theorem, which is purely topological, is given in Section 1. In Section 2, I will prove a theorem on diophantine approximation which is needed to fill a gap in the Bianchi–Humbert theory. This gap is due to the fact that Bianchi and Humbert consider only quadratic forms with coefficients in the given quadratic field and not with arbitrary complex coefficients. An account of the Bianchi–Humbert theory is given in Section 3. In Section 4, this theory is combined with the theorem of Section 1 to yield the required presentation.

The only real difficulty remaining is the problem of determining the fundamental domain for the Bianchi–Humbert theory. This is considered in Sections 7–9. For small values of m we can refer to Bianchi's paper [2] for the equations defining the fundamental domains. In the remaining cases, the calculation is best done by a machine. The results of Sections 7–9 are sufficiently explicit to make this possible but it has not yet been done.

After this paper was written, I received a copy of the paper "The singular points of the fundamental domains for the groups of Bianchi" by W. M. Woodruff. In this paper, the main results of Section 3, as well as many other interesting results concerning the fundamental domains, were obtained independently. Woodruff's treatment is closer to that of Bianchi, and to classical automorphic function theory, than the one given here which is based on the work of Humbert [10].

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Another method of deducing a presentation of $SL(2, Z[\sqrt{-5}])$ from Bianchi's work was discovered by Serre. It uses Bianchi's "extended" group. It would be interesting to know if this method can be generalized. The main point is that in the cases considered by Bianchi, the extended group is generated by reflections. Humbert [10] doubted that Bianchi's results could be extended to all quadratic imaginary fields but as far as I know, no further work has been done on this.[1]

A completely different method for finding presentations of $SL(2, \mathcal{O})$ in the Euclidean case was found by P. M. Cohn [7]. This is much simpler than the method used here. In the cases where $\mathcal{O}$ is not Euclidean, Cohn's method yields a presentation of the elementary subgroup $E(2, \mathcal{O})$ rather than $SL(2, \mathcal{O})$. He also shows that $E(2, \mathcal{O}) \neq SL(2, \mathcal{O})$ in the case where $\mathcal{O}$ is not Euclidean [6]. This is not at all clear from the point of view of the present paper.

1. Transformation Groups

Let G be a group acting on a topological space X (not necessarily faithfully). Let V be an open set of X such that the sets σV for $\sigma \in G$ cover X, i.e., $GV = X$. Let S be the set of $\sigma \in G$ such that $\sigma V \cap V \neq \emptyset$. As in [13] we define a group Γ as follows. Γ has one generator $[\sigma]$ for each $\sigma \in S$ and relations $[\sigma\tau] = [\sigma][\tau]$ for each pair σ, τ such that $V \cap \sigma V \cap \sigma\tau V \neq \emptyset$. As pointed out in [13] Γ can also be defined as the *monoid* defined by these generators and relations because $1 \in S$, $\sigma \in S$ implies $\sigma^{-1} \in S$, and the relations include the relations $[\sigma] = [\sigma][1]$, $[\tau] = [1][\tau]$, and $[\sigma][\sigma^{-1}] = [1]$. Thus $[1]$ is the unit element and $[\sigma^{-1}]$ is the inverse of $[\sigma]$.

Define a group homomorphism $\epsilon : \Gamma \to G$ by $\epsilon([\sigma]) = \sigma$. It is easy to see [13] that ϵ is onto if X is connected.

In order to state the main theorem of this section we need one more definition.

Definition. We say an element $\alpha \in \pi_1(X, x)$ is represented by a loop in $A \subset X$ if there is a path ω from x to a point $a \in A$ and a loop l in A based at a so that $\omega l \omega^{-1}$ represents α.

The product $\omega l \omega^{-1}$ denotes the usual composition of paths. Clearly, the elements represented by a loop in A form a subset of $\pi_1(X, x)$ stable under all inner automorphisms.

[1] Mennicke has recently shown that the extended Bianchi group is not generated by reflections in general.

THEOREM 1.1. *Let X be a pathwise connected space and V a pathwise connected open subset. Let G be a group acting on X such that $GV = X$. Let Γ be the group constructed above. Then there is an exact sequence of group homomorphisms*

$$1 \longrightarrow N \longrightarrow \pi_1(X) \xrightarrow{\theta} \Gamma \xrightarrow{\epsilon} G \longrightarrow 1,$$

where N is the normal subgroup of $\pi_1(X)$ generated by all elements represented by a loop in at least one of the sets $\sigma V \cup \tau V$, $\tau, \sigma \in G$.

The main theorem of [13] follows from this if $\pi_1(X) = 0$. The method of proof used here follows the same general lines as that in [13]. The definition of θ will be given in the proof of the theorem. It depends on the choice of a base point in V but any two choices for θ will differ by an inner automorphism of $\pi_1(X)$.

Remark. The definition of N given in [16] is not quite correct. If we use only the sets $V \cup \sigma V$, N must be defined as the smallest G-stable subgroup of $\pi_1(X)$ containing all elements represented by a loop on one of these sets. The precise definition of this is given below.

Before giving the proof, I will make some remarks about the action of G. If H is any group, let $\operatorname{Aut} H$ be the group of automorphisms of H, $\operatorname{In} H \subset \operatorname{Aut} H$ the normal subgroup of inner automorphisms, and let $\operatorname{Out} H = \operatorname{Aut} H/\operatorname{In} H$. An outer action of G on H will mean a homomorphism $\rho : G \to \operatorname{Out} H$. Thus each $\sigma \in G$ determines an automorphism $\rho(\sigma)$ of H unique up to inner automorphisms. It makes sense to say that a *normal* subgroup N of H is stable under the action of G since this is equivalent to the stability of N under the subgroup of $\operatorname{Aut} H$ which is the inverse image of $\rho(G)$. If we are given outer G actions ρ, η on H and K, we say that a homomorphism $f : H \to K$ is an outer G-homomorphism if for each $\sigma \in G$ the diagram

$$\begin{tikzcd} H \arrow[r, "f"] \arrow[d, "\rho(\sigma)"'] & H \arrow[d, "\eta(\sigma)"] \\ H \arrow[r, "f"] & K \end{tikzcd}$$

commutes up to an inner automorphism of K. This is clearly independent of the choice of $\rho(\sigma)$, $\eta(\sigma)$ modulo inner automorphisms.

As an example, if $G = \Gamma/H$ is a quotient group of Γ, the action of Γ on H by conjugation gives a homomorphism $\Gamma \to \operatorname{Aut} H$ sending H onto

In H. Therefore we get an outer action $G = \Gamma/H \to \operatorname{Out} H$ of G on H. Thus in Theorem 2 there is an outer action of G on $\ker \epsilon$.

If G acts on a pathwise connected space X, each $\sigma \in G$ induces a homomorphism $\sigma_* : \pi_1(X) \to \pi_1(X)$ unique up to inner automorphism. This is composed of the unique $\sigma_* : (X, x) \to \pi_1(X, \sigma x)$ followed by the nonunique $\pi_1(X, \sigma x) \to \pi_1(X, x)$ obtained by choosing a path ω from x to σx and sending each loop l at σx to $\omega l \omega^{-1}$. This clearly defines an outer action of G on $\pi_1(X)$.

ADDENDUM TO THEOREM 1.1. *The map* $\theta : \pi_1(X) \to \ker \epsilon$ *is an outer* G*-homomorphism.*

Using this we can give a reasonable presentation for G. We say that a set of elements $x_1, x_2, \ldots \in \pi_1(X)$ generates $\pi_1(X)$ normally over G if the smallest G-stable normal subgroup of $\pi_1(X)$ containing the given set is $\pi_1(X)$ itself.

COROLLARY 1.2. *Let* $\alpha_\nu \in \pi_1(X)$ *be elements which generate* $\pi_1(X)$ *normally over* G. *Then* $\ker \epsilon$ *is the smallest normal subgroup of* Γ *containing the elements* $\theta(\alpha_\nu)$.

This is an immediate consequence of the Theorem 1.1 and its addendum because the smallest normal subgroup of Γ containing the $\theta(\alpha_i)$ is obviously the same as the smallest G-stable normal subgroup of $\ker \epsilon$ containing the $\theta(\alpha_i)$ but the property of normal generation over G is clearly preserved by outer G homomorphisms (onto).

We now turn to the proof of Theorem 1.1. Let $\omega : I \to X$ be any path in X. Suppose we are given $\tau, \sigma \in G$ so that $\omega(0) \in \sigma V$, $\omega(1) \in \tau(V)$. Define an element $\theta(\sigma, \omega, \tau) \in \Gamma$ as follows. Since the ρV, $\rho \in G$ cover X we can find a subdivision of I given by $0 = t_0 < t_1 < \cdots < t_n = 1$ so that for each i, if $I_i = [t_i, t_{i+1}]$, we have $\omega(I_i) \subset \rho_i V$ for some $\rho_i \in G$. We can further assume that $\omega(I_0) \subset \sigma V$, $\omega(I_{n-1}) \subset \tau V$ and choose $\rho_0 = \sigma$, $\rho_{n-1} = \tau$. Since $\omega(t_i) \in \omega(I_{i-1}) \cap \omega(I_i)$, we have $\rho_{i-1} V \cap \rho_i V \neq \emptyset$ for $1 \leqslant i \leqslant n-1$. Therefore $\rho_{i-1}^{-1}\rho_i \in S = \{\rho \in G \mid V \cap \rho V \neq \emptyset\}$. Define

$$\theta(\sigma, \omega, \tau) = [\rho_0^{-1}\rho_1][\rho_1^{-1}\rho_2] \cdots [\rho_{n-2}^{-1}\rho_{n-1}] \tag{1}$$

LEMMA 1.3. $\theta(\sigma, \omega, \tau)$ *is well-defined*, i.e., *independent of the subdivision used and of the choice of the* ρ_i.

Proof. Suppose we replace some ρ_i, $1 \leqslant i \leqslant n-2$, by another ρ_i' with $\omega(I_i) \subset \rho_i' V$. This changes $\theta(\sigma, \omega, \tau)$ by replacing $[\rho_{i-1}^{-1}\rho_i]$

$[\rho_i^{-1}\rho_{i+1}]$ by $[\rho_{i-1}^{-1}\rho_i'][\rho_i'^{-1}\rho_{i+1}]$. Now $\omega(t_{i-1}) \in \rho_{i-1}V \cap \rho_i V \cap \rho_i' V$ so $V \cap \rho_{i-1}^{-1}\rho_i V \cap \rho_{i-1}^{-1}\rho_i' V \neq \emptyset$. Similarly $\omega(t_i) \in \rho_i' V \cap \rho_i V \cap \rho_{i+1} V$ so $V \cap \rho_i'^{-1}\rho_i V \cap \rho_i'^{-1}\rho_{i+1} V \neq \emptyset$. Therefore, in Γ, we have $[\rho_{i-1}^{-1}\rho_i'] = [\rho_{i-1}^{-1}\rho_i][\rho_i^{-1}\rho_i']$ and $[\rho_i'^{-1}\rho_{i+1}] = [\rho_i'^{-1}\rho_i][\rho_i^{-1}\rho_{i+1}]$ so

$$[\rho_{i-1}^{-1}\rho_i'][\rho_i'^{-1}\rho_{i+1}] = [\rho_{i-1}^{-1}\rho_i][\rho_i^{-1}\rho_i'][\rho_i'^{-1}\rho_i][\rho_i^{-1}\rho_{i+1}] = [\rho_{i-1}^{-1}\rho_i][\rho_i^{-1}\rho_{i+1}]$$

using the relation $[\rho_i^{-1}\rho_i'][\rho_i'^{-1}\rho_i] = 1$. This shows that θ is independent of the choice of the ρ_i.

Since any two subdivisions of I have a common refinement, it will suffice to check that $\theta(\sigma, \omega, \tau)$ is unchanged by an elementary subdivision in which one of the I_i is divided into two parts I_i', I_i'' by some t with $t_i < t < t_{i+1}$. If $\omega(I_i) \subset \rho_i V$, we may choose this same ρ_i for I_i' and I_i''. There is no difficulty even if $i = 0$ or n. Thus, in the expression (1) for $\theta(\sigma, \omega, \tau)$, the only change is that $[\rho_i^{-1}\rho_{i+1}]$ is replaced by $[\rho_i^{-1}\rho_i][\rho_i^{-1}\rho_{i+1}]$. But $[\rho_i^{-1}\rho_i] = [1] = 1$ in Γ. This proves the lemma.

From the definition of θ we obtain immediately the following useful properties

$$\theta(\rho, \omega, \sigma)\, \theta(\sigma, \omega', \tau) = \theta(\rho, \omega\omega', \tau) \tag{2}$$

$$\theta(\sigma, \omega, \tau)^{-1} = \theta(\tau, \omega^{-1}, \sigma) \tag{3}$$

$$\theta(\sigma, \hat{x}, \sigma) = 1. \tag{4}$$

Here $\hat{x}$ denotes the trivial path sending all of I to the point x and ω^{-1} is the inverse path $\omega^{-1}(t) = \omega(1 - t)$. To check (2) we merely subdivide ω and ω' and put the two subdivisions together to subdivide $\omega\omega'$. For (3), use the same subdivision for ω and ω^{-1}.

LEMMA 1.4. *If ω and ω' are homotopic with fixed endpoints, then $\theta(\sigma, \omega, \tau) = \theta(\sigma, \omega', \tau)$.*

Proof. If $f : I \to I$ is a homeomorphism with $f(0) = 0$, $f(1) = 1$, then $\theta(\sigma, \omega, \tau) = \theta(\sigma, \omega \circ f, \tau)$ by the definition of θ. Now suppose $\omega \simeq \omega'$ with fixed endpoints. Then there is a map $g : I \times I \to X$ so $g(0, t) = \omega(0) = \omega'(0)$ for all t, $g(1, t) = \omega(1) = \omega'(1)$, $g(t, 0) = \omega(t)$, and $g(t, 1) = \omega'(t)$. Divide $I \times I$ into smaller squares by lines parallel to the sides of $I \times I$. If these are small enough, we can assume that for each small square Q we have $g(Q) \subset \rho V$ for some $\rho \in G$ and also that we can take $\rho = \sigma$ if Q touches the side $\{0\} \times I$ and $\rho = \tau$ if Q touches the side $\{1\} \times I$. Let $p : I \to I \times I$ be the path $p(t) = (t, 0)$ and let $p' : I \to I \times I$ be the polygonal path formed by the 3 sides $\{0\} \times I$,

$I \times \{1\}$, and $\{1\} \times I$. Then $\omega = gp$ and $\omega'' = gp'$ is, up to a homeomorphism of I, the composition $\widehat{\omega(0)}\, \omega' \widehat{\omega(1)}$ where $\widehat{\omega(0)}$, $\widehat{\omega(1)}$ denote trivial paths. By properties (2) and (4) above, we see that $\theta(\sigma, \omega'', \tau) = \theta(\sigma, \widehat{\omega(0)}, \sigma)\, \theta(\sigma, \omega', \tau)\, \theta(\sigma, \widehat{\omega(1)}, \tau) = \theta(\sigma, \omega, \tau)$. Now p' can be deformed into p by a series of elementary deformations in which a path composed of one or more sides of a small square Q is replaced by the path composed of the remaining sides of Q [13]. Suppose $q, q' : I \to I \times I$ are two polygonal paths differing only by an elementary deformation over the small square Q. Subdivide I into subintervals $I_0, \ldots, I_{n-1}$ so that each $q(I_i)$ is contained in some small square and such that $q \mid I_\nu$ is that part of q which is deformed over Q. Thus $q' \mid I_i = q \mid I_i$ for $i \neq \nu$ and $q(I_\nu)$, $q'(I_\nu)$ both lie in Q. We may use this subdivision to calculate both $\theta(\sigma, gq, \tau)$ and $\theta(\sigma, gq', \tau)$. We may also use the same ρ_i for both gq and gq'. This is clear for $i \neq \nu$. For ρ_ν in both cases, we may use some $\rho \in G$ such that $g(Q) \subset \rho V$. Of course, we choose $\rho = \sigma$ if Q meets $\{0\} \times I$ and $\rho = \tau$ if Q meets $\{1\} \times I$. This shows $\theta(\sigma, gq, \tau) = \theta(\sigma, gq', \tau)$ and so, by induction on the number of elementary deformations, $\theta(\sigma, \omega, \tau) = \theta(\sigma, \omega', \tau)$.

Definition. Choose a base point $x \in V$ and define $\theta : \pi_1(X, x) \to \Gamma$ by $\theta(\omega) = \theta(1, \omega, 1)$ for each loop ω at x.

By Lemma 1.4 and properties (2) and (4) we see that θ is a well-defined homomorphism. By the definition (1) of θ we see

$$\epsilon\theta(\sigma, \omega, \tau) = \sigma^{-1}\tau. \tag{5}$$

Therefore θ maps $\pi_1(X, x)$ into the kernel of ϵ.

We now check that θ is an outer G-homomorphism. Let $\rho \in G$. Then

$$\theta(\rho\sigma, \rho\omega, \rho\tau) = \theta(\sigma, \omega, \tau). \tag{6}$$

In fact, if $\omega(I_i) \subset \rho_i V$, then $\rho\omega(I_i) \subset \rho\rho_i V$ so we can use the same subdivision for ω and $\rho\omega$. But $(\rho\rho_i)^{-1} \rho\rho_{i+1} = \rho_i^{-1}\rho_{i+1}$ so (6) follows from (1). Now, the action of ρ on $\pi_1(X)$ is defined as follows. Choose a path λ from the base point x to ρx. Then, if ω is a loop at x, the action of ρ sends ω to $\lambda\rho(\omega)\,\lambda^{-1}$. Now $\theta(\lambda\rho(\omega)\,\lambda^{-1}) = \theta(1, \lambda, \rho)\, \theta(\rho, \rho\omega, \rho)\, \theta(\rho, \lambda^{-1}, 1) = \theta(1, \lambda, \rho)\, \theta(\omega)\, \theta(1, \lambda, \rho)^{-1}$ using (2), (3), and (6). By (5), $\epsilon\theta(1, \lambda, \rho) = \rho$ so conjugation by $\theta(1, \lambda, \rho)$ defines the outer action of ρ on ker ϵ. This proves our assertion.

We next show that the subgroup N of $\pi_1(X)$ lies in the kernel of θ.

Let $\alpha \in \pi_1(X)$ be represented by a loop in $\sigma V \cup \tau V$. Then we can find a point $y \in \sigma V \cup \tau V$, say $y \in \sigma V$, a path λ from x to y, and a loop ω in $\sigma V \cup \tau V$ based at y, so that $\lambda\omega\lambda^{-1}$ represents α. Now

$$\theta(\alpha) = \theta(1, \lambda, \sigma)\, \theta(\sigma, \omega, \sigma)\, \theta(\sigma, \lambda^{-1}, 1) = \theta(1, \lambda, \sigma)\, \theta(1, \sigma^{-1}\omega, 1)\, \theta(1, \lambda, \sigma)^{-1}$$

by (2), (3), and (6). Therefore it will suffice to show that $\theta(1, \sigma^{-1}\omega, 1) = 1$. Let $\omega' = \sigma^{-1}\omega$ and $\rho = \sigma^{-1}\tau$. Then ω' is a loop in $V \cup \rho V$. Subdivide I into intervals such that $\omega'(I_i)$ is alternately contained in V and ρV. Then $\theta(1, \omega', 1) = [1^{-1}\rho][\rho^{-1}1] \cdots [1^{-1}\rho][\rho^{-1}1] = 1$ since $[\rho^{-1}] = [\rho]^{-1}$. Of course, if $\omega'(I) \subset V$ we have $\theta(1, \omega', 1) = 1$ immediately.

To prove Theorem 1.1, we construct a map $\varphi : \ker \epsilon \to \pi_1(X)/N$ which is an inverse for the map $\theta' : \pi_1(X)/N \to \ker \epsilon$ induced by θ. Let Γ' be the free monoid on the symbols $[\sigma]$ for $\sigma \in S$. Define $\epsilon' : \Gamma' \to G$ by $\epsilon([\sigma]) = \sigma$. Then the canonical map $\Gamma' \to \Gamma$ (by $[\sigma] \to [\sigma]$) maps $\ker \epsilon'$ onto $\ker \epsilon$. We first define $\varphi' : \ker \epsilon' \to \pi_1(X)/N$. Let $\alpha = [\sigma_1] \cdots [\sigma_n] \in \ker \epsilon'$ where $\sigma_i \in S$. Since $\epsilon(\alpha) = 1$, $\sigma_1 \cdots \sigma_n = 1$ in G. Let $\rho_0 = 1$, $\rho_i = \sigma_1 \cdots \sigma_i$ for $1 \leqslant i \leqslant n$. Note that $\rho_n = 1$. Since all $\sigma_i \in S$, we have $\rho_{i-1}V \cap \rho_i V = \rho_{i-1}(V \cap \sigma_i V) \neq \emptyset$ for $1 \leqslant i \leqslant n$. Choose a point $x_i \in \rho_{i-1}V \cap \rho_i V$ for $1 \leqslant i \leqslant n$ and let $x_0 = x_{n+1} = x$, the base point. Choose a path ω_i from x_i to x_{i+1} in $\rho_i V$ for $0 \leqslant i \leqslant n$. Then $\omega_0\omega_1 \cdots \omega_n$ is a loop ω based at x. We let $\varphi'(\alpha)$ be the homotopy class of ω (mod N).

LEMMA 1.5. *The map $\varphi' : \ker \epsilon' \to \pi_1(X)/N$ is well-defined.*

Proof. Suppose we change a point x_i, $1 \leqslant i \leqslant n$, and the paths ω_{i-1}, ω_i to $x_i', \omega_{i-1}', \omega_i'$, respectively. Then we get a loop $\omega' = \omega_0 \cdots \omega_{i-1}'\omega_i'\omega_{i+1} \cdots \omega_n$. Now

$$\omega'\omega^{-1} \simeq \omega_0 \cdots \omega_{n-2}\omega_{i-1}'\omega_i'\omega_i^{-1}\omega_{i-1}^{-1}(\omega_0 \cdots \omega_{n-2})^{-1}.$$

But $\omega_{i-1}'\omega_i'\omega_i^{-1}\omega_{i-1}^{-1}$ is a loop in $\rho_{i-1}V \cap \rho_i V$ based at x_{i-1}. Therefore $\omega'\omega^{-1}$ represents an element of N. Clearly, any change of the x_i and ω_i can be obtained by a succession of changes of the type just considered.

LEMMA 1.6. *The map φ' factors uniquely into $\varphi\eta'$ where $\eta' : \ker \epsilon' \to \ker \epsilon$ is the restriction of the canonical map $\eta : \Gamma' \to \Gamma$ and $\varphi : \ker \epsilon \to \pi_1(X)/N$.*

Proof. We have already observed, following [13], that Γ may be regarded as the *monoid* generated by the $[\sigma]$, $\sigma \in S$, with relations

$[\sigma\tau] = [\sigma][\tau]$ for $V \cap \sigma V \cap \sigma\tau V \neq \emptyset$. Therefore two words in Γ' have the same image in Γ if and only if one can be transformed into the other by a series of elementary transformations obtained by replacing $[\sigma\tau]$ by $[\sigma][\tau]$ or $[\sigma][\tau]$ by $[\sigma\tau]$ for various σ, τ satisfying $V \cap \sigma V \cap \sigma\tau V \neq \emptyset$. It will clearly suffice to show that φ' takes the same value for two words differing by such an elementary transformation.

Let $\alpha = [\sigma_1] \cdots [\sigma_n]$ and $\alpha' = [\sigma_1'] \cdots [\sigma_{n+1}']$ where $\sigma_i' = \sigma_i$ for $i < j$, $\sigma_i' = \sigma_{i-1}$ for $i > j+1$, and $\sigma_j'\sigma_{j+1}' = \sigma_j$ with $V \cap \sigma_j' V \cap \sigma_j V \neq \emptyset$. Let $\rho_i = \sigma_1 \cdots \sigma_i$, $\rho_i' = \sigma_1' \cdots \sigma_i'$. Then $\rho_i' = \rho_i$ for $i < j$, $\rho_i' = \rho_{i-1}$ for $i > j$ and $\rho_j' = \rho_{j-1}\sigma_i'$. Applying the definition of θ' we choose points x_i and paths $\omega_0, \ldots, \omega_n$ for α and points x_i' and paths $\omega_0', \ldots, \omega_{n+1}'$ for α_i'. We can clearly choose $x_i' = x_i \in \rho_{i-1}V \cap \rho_i V = \rho_{i-1}' V \cap \rho_i' V$ for $i < j$ and $x_i' = x_{i-1} \in \rho_{i-2} V \cap \rho_{i-1} V = \rho_{i-1}' V \cap \rho_i' V$ for $i > j+1$. Since $V \cap \sigma_j' V \cap \sigma_j V \neq \emptyset$, we have $\rho_{j-1} V \cap \rho_j' V \cap \rho_j V \neq \emptyset$. Choose a point y in this set and let $x_j = x_j' = x_{j+1}' = y$. Note $\rho_j = \rho_{j+1}'$ and $\rho_{j-1}' = \rho_{j-1}$. We can now choose $\omega_i' = \omega_i$ for $i < j$ and $\omega_i' = \omega_{i-1}$ for $i > j$. Choose ω_j' to be the trivial path at $x_j' = x_{j+1}'$. Then $\varphi'(\alpha') = \omega_0' \cdots \omega_{n+1}' = \omega_0 \cdots \omega_{j-1}\omega_j'\omega_j \cdots \omega_n \simeq \omega_0 \cdots \omega_{j-1}\omega_j \cdots \omega_n = \varphi'(\alpha)$. This proves the lemma.

We can now finish the proof of Theorem 1.1. It will suffice to show that $\theta'\varphi$ and $\varphi\theta'$ are identity maps. For $\varphi\theta'$, let ω be a loop based at x. Subdivide I into subintervals $I_i = [t_i, t_{i+1}]$, $i = 0, \ldots, n$ with $\omega(I_i) \subset \rho_i V$, $\rho_0 = 1 = \rho_n$. Let $\sigma_i = \rho_{i-1}^{-1}\rho_i$ for $i = 1, \ldots, n$. Then $\theta(\omega) = [\sigma_1] \cdots [\sigma_n]$. Now $\sigma_1 \cdots \sigma_i = \rho_i$. To calculate $\varphi\theta(\omega)$ we may choose $x_i = \omega(t_i) \in \rho_{i-1} V \cap \rho_i V$ and choose $\omega_i = \omega \mid I_i$. Therefore $\varphi\theta(\omega)$ is represented by $\omega_0 \cdots \omega_n \simeq \omega$.

For $\theta'\varphi$, let $\alpha = [\sigma_1] \cdots [\sigma_n]$, $\rho_i = \sigma_1 \cdots \sigma_i$, choose $x_i \in \rho_{i-1} V \cap \rho_i V$, $x_0 = x_{n+1} = x$ and choose ω_i joining x_i to x_{i+1} in $\rho_i V$. Then $\varphi(\alpha)$ is represented by the loop $\omega = \omega_0 \cdots \omega_n$. To calculate $\theta(\omega)$ we may subdivide ω into the same paths $\omega_0, \ldots, \omega_n$ and choose $\rho_i V$ containing ω_i. Then $\theta(\omega) = [\rho_0^{-1}\rho_1] \cdots [\rho_{n-1}^{-1}\rho_n] = [\sigma_1] \cdots [\sigma_n] = \alpha$. This finishes the proof of Theorem 1.1.

We now make some comments on the case where we have a *closed* set A with $GA = X$.

DEFINITION. A subset A of X is called G-inflatable if there is an open set $V \supset A$ such that $\sigma_1 V \cap \cdots \cap \sigma_n V \neq \emptyset$ if and only if $\sigma_1 A \cap \cdots \cap \sigma_n A \neq \emptyset$.

If X is locally pathwise connected and A is connected, we may replace V by the component of V containing A. This is open and pathwise connected. If $GA = X$, then $GV = X$ and we can apply Corollary 1.2 to get a presentation of G. The group Γ is generated by the $[\sigma]$ with $A \cap \sigma A \neq \emptyset$ with relations $[\sigma\tau] = [\sigma][\tau]$ for $A \cap \sigma A \cap \sigma\tau A \neq \emptyset$. We must now calculate the map θ.

Suppose $\lambda : S^1 \to X$ is any loop in X, (S^1 being the circle). Let ω be a path from the base point x to some point y on $\lambda(S^1)$. Choose some $s \in S^1$ such that $\lambda(s) = y$. Using s as base point for S^1 we can define $\omega\lambda\omega^{-1}$. This represents an element α of $\pi_1(X)$ determined up to conjugacy. We say that λ represents α up to conjugacy. If $y \in \sigma A \subset \sigma V$ we have $\theta(\omega\lambda\omega^{-1}) = \theta(1, \omega, \sigma)\, \theta(\sigma, \lambda, \sigma)\, \theta(\sigma, \omega^{-1}, 1)$. We define $\theta(\lambda) = \theta(\sigma, \lambda, \sigma)$. This is not unique but is determined up to conjugacy in Γ. If $\rho \in G$, (6) shows that $\theta(\rho\lambda)$ is conjugate to $\rho\theta(\lambda)$, the image of $\theta(\lambda)$ under the outer action of ρ on ker ϵ.

Definition. The loop $\lambda : S^1 \to X$ is called well-behaved (relative to A) if there is a subdivision of S^1 into arcs $I_0, \ldots, I_n$ such that for each i there is some $\rho_i \in G$ with $\lambda(I_i) \subset \rho_i A$.

For such a loop, $\theta(\lambda)$ is clearly conjugate to

$$[\rho_0^{-1}\rho_1][\rho_1^{-1}\rho_2] \cdots [\rho_{n-1}^{-1}\rho_n][\rho_n^{-1}\rho_0].$$

To see this, we choose the base point s inside I_0, dividing I_0 into arcs I_0', I_0''. Now $I_0' \subset \rho_0 V$, $I_i \subset \rho_i V$, $I_0'' \subset \rho_0 V$ and the definition of $\theta(\rho_0, \lambda, \rho_0)$ gives the required result.

Corollary 1.7. *Let X be a connected, locally pathwise connected space. Let G be a group acting on X. Let A be a G-inflatable connected subset of X such that $GA = X$. Let $\alpha_i \in \pi_1(X)$ be elements which generate $\pi_1(X)$ normally over G. Let λ_i be well-behaved loops relative to A such that λ_i represents α_i up to conjugacy. Let S be the set of $\sigma \in G$ with $\sigma A \cap A \neq \emptyset$. Then G has a presentation with generators $[\sigma]$ for $\sigma \in S$ and relations $[\sigma\tau] = [\sigma][\tau]$ for $A \cap \sigma A \cap \sigma\tau A \neq \emptyset$ and also the relations $\theta(\lambda_i) = 1$.*

Recall that a closed subset A of X is called G-normal [13] if each point of X has a neighborhood U such that $U \cap \sigma A \neq \emptyset$ for only a finite number of $\sigma \in G$. Theorem 2 of [13] asserts that if X is a metric space and G acts on X by isometries then any closed G-normal subset A of X is G-inflatable. Further examples of G-inflatable sets may be obtained from the following lemma.

Lemma 1.8. *Let D be a G-inflatable subset of X. Let E be a nonempty subset of G. Let A be the union of all eD for $e \in E$. Then A is G-inflatable.*

Proof. Let $V \supset D$ be open such that $\sigma_1 V \cap \cdots \cap \sigma_n V \neq \emptyset$ if and only if $\sigma_1 D \cap \cdots \cap \sigma_n D \neq \emptyset$. Let U be the union of all eV for $e \in E$. Then U is open and contains A. If $\bigcap \sigma_i U \neq \emptyset$, there is some $y \in X$ such that $y \in \sigma_i U$ for all i. Therefore we can find $e_i \in E$ so $y \in \sigma_i e_i V$ so $\bigcap \sigma_i e_i V \neq \emptyset$. By the choice of V, $\bigcap \sigma_i e_i D \neq \emptyset$ so $\bigcap \sigma_i A \neq \emptyset$.

We now determine more explicitly the presentation of the group Γ in a special case which will be of interest to us later. We assume that an intersection $\sigma_1 A \cap \sigma_2 A \cap \sigma_3 A$ is empty unless at least two of the sets $\sigma_1 A$, $\sigma_2 A$, $\sigma_3 A$ are the same. Let H be the set of all $\sigma \in G$ such that $\sigma A = A$. Then H is a subgroup of G. The set S of σ such that $\sigma A \cap A \neq \emptyset$ is clearly stable under left and right multiplication by elements of H. Therefore S is a disjoint union of double cosets $H\sigma_\nu H$. We may choose $\sigma_0 = 1$ since $H \subset S$. Note $\sigma_\nu^{-1} \in S$ also so $\sigma_\nu = h_\nu \sigma_{\mu_\nu} h_\nu'$ for some h_ν, $h_\nu' \in H$ and μ_ν.

LEMMA 1.9. *Suppose $\sigma_1 A \cap \sigma_2 A \cap \sigma_3 A \neq \emptyset$ implies that at least two of the sets $\sigma_1 A$, $\sigma_2 A$, $\sigma_3 A$ are equal. Then, with the above notations, Γ is generated by all $[h]$ with $h \in H$ and all $[\sigma_\nu]$ for $\nu \neq 0$ with the following relations.*

(a) *All $[hh'] = [h][h']$ for $h, h' \in H$;*
(b) *The relations $[\sigma_\nu]^{-1} = [h_\nu][\sigma_{\mu_\nu}][h_\nu']$ where $\sigma_\nu^{-1} = h_\nu \sigma_{\mu_\nu} h_\nu'$ in G;*
(c) *All relations $[\sigma_\nu] = [h][\sigma_\nu][h']$ with $\sigma_\nu = h\sigma_\nu h'$ in G.*

Proof. It follows immediately from the hypothesis that

$$A \cap \sigma A \cap \sigma\tau A \neq \emptyset$$

if and only if at least one of σ, τ, $\sigma\tau$ lie in H and the remaining ones lie in S. If $\sigma, \tau \in H$, we get the relations (a). If $\sigma = h \in H$ $\tau = h_1 \sigma_\nu h_2$, $\sigma\tau = h_3 \sigma_\mu h_4$, then $hh_1\sigma_\nu h_2 = h_3 \sigma_\mu h_4$, so $\mu = \nu$, and we get the relation $[h][h_1\sigma_\nu h_2] = [h_3\sigma_\nu h_4]$. Similarly from $\tau = h \in H$ we get the relations $[h_1\sigma_\nu h_2][h] = [h_3\sigma_\nu h_4]$ whenever $h_1\sigma_\nu h_2 h = h_3 \sigma_\nu h_4$. In particular, we have $[h_1\sigma_\nu h_2] = [h_1][\sigma_\nu][h_2]$, showing that Γ is generated by the $[h]$, $h \in H$ and the $[\sigma_\nu]$, $\nu \neq 0$. We also see that $\sigma_\nu = h\sigma_\nu h'$ implies $[\sigma_\nu] = [h][\sigma_\nu][h']$, i.e., the relations (c) hold in Γ. We now eliminate the generator $[h\sigma_\nu h']$ by choosing a representation for it in terms of our chosen generators, i.e., $[h\sigma_\nu h'] = [h][\sigma_\nu][h']$. This representation is unambiguous modulo the relations (a) and (c) because if $h\sigma_\nu h' = h_1\sigma_\nu h_1'$, then $\sigma_\nu = h^{-1}h_1\sigma_\nu h_1' h'^{-1}$ so $[\sigma_\nu] = [h]^{-1}[h_1][\sigma_\nu][h_1'][h']^{-1}$. Furthermore, the relations obtained above follow automatically from the relations (a) and (c) and the representation $[h\sigma_\nu h'] = [h][\sigma_\nu][h']$. Finally, consider the case $\sigma\tau \in H$ with $\sigma = h_1\sigma_\nu h_2$, $\tau = h_3\sigma_\mu h_4$, $\sigma\tau = h_5 \in H$. This gives the relation $[h_1\sigma_\nu h_2][h_3\sigma_\mu h_4] = [h_5]$ or $[h_1][\sigma_\nu][h_2h_3][\sigma_\mu][h_4] = [h_5]$, or $[\sigma_\nu]^{-1} = [h][\sigma_\mu][h']$ where $h = h_2h_3$, $h' = h_4h_5^{-1}h_1$ and $\sigma_\nu^{-1} = h\sigma_\mu h'$. This gives the relations (b). If we have two different choices $\sigma_\nu^{-1} = h\sigma_\mu h' = h_1\sigma_\mu h_1'$, the two relations we obtain are clearly equivalent modulo the relations (a) and (c).

Remark. In the applications given here we make no use of the part of Theorem 1.1 which specifies the kernel N of θ. This may be useful in computing the fundamental group of certain spaces with groups of transformations. For example, we can immediately calculate $\pi_1(S^1)$ by taking G to be a cyclic rotation group of order at least 3. The same example with order 2 shows that it is necessary to consider all sets $\sigma V \cup \tau V$ in defining N and not just the sets σV. Of course, if all $\sigma V \cap \tau V$ are pathwise connected, van Kampen's theorem shows that N is the normal subgroup of $\pi_1(X)$ generated by the images of all $\pi_1(\sigma V)$ or, equivalently the smallest G-stable normal subgroup of $\pi_1(X)$ containing the image of $\pi_1(V) \to \pi_1(X)$. In general, N is the smallest G-stable normal subgroup of $\pi_1(X)$ containing all images $\pi_1(V \cup \sigma V) \to \pi_1(X)$.

In order to obtain a presentation of G, we must find a set of generators for $\theta(\pi_1(X))$ as a normal subgroup of Γ. It will obviously suffice to find a set of normal generators for $\pi_1(X)$ and apply θ. In §4, we will do this using the following result. This result is, of course, well-known and classical but I will include a proof for the reader's convenience.

LEMMA 1.10. *Let M be a combinatorial n-manifold. Let K be a subcomplex of M of dimension $\leqslant n-2$. For each $n-2$ cell e_i of K, choose a small loop λ_i around e_i and let $x_i \in \pi_1(M-K)$ be represented (up to conjugacy) by λ_i. Let N be the normal subgroup of $\pi_1(M-K)$ which is generated normally by the x_i. Then the sequence*

$$1 \to N \to \pi_1(M-K) \to \pi_1 M \to 1$$

is exact.

Proof. By starting with K and removing one maximal cell at a time, we produce a collection of subcomplexes K_α of K such that $K - K_\alpha$ is finite for each α. If we order the α by $\alpha \leqslant \beta$ if $K_\alpha \supset K_\beta$, we obtain a directed set. Since $M = \bigcup (M - K_\alpha)$, we have $\pi_1(M) = \varinjlim \pi_1(M - K_\alpha)$. If N_α is the normal subgroup of $\pi_1(M-K)$ generated normally by the

x_i for which $e_i \not\subset K_\alpha$, then $N = \varinjlim N_\alpha$. Therefore, it will suffice to show that the sequence

$$1 \to N_\alpha \to \pi_1(M - K) \to \pi_1(M - K_\alpha) \to 1$$

is exact. By induction on the number of cells in $K - K_\alpha$, it will be enough to show that if $K_\alpha = K_\beta \cup e$ where the cell e is of dimension k, then $\pi_1(M - K_\alpha) = \pi_1(M - K_\beta)$ for $k < n - 2$, while $\pi_1(M - K_\beta) = \pi_1(M - K_\alpha)/L$ for $k = n - 2$, L being the normal subgroup generated by a small loop about e.

Let $X = K - K_\beta$ and $Y = K - K_\alpha$. Choose a neighborhood of e of the form $N = D^k \times D^{n-k}$ where D^k is a k-disk, such that $e = D^k \times \{0\}$. Then $X = Y \cup N$ while $Y \cap N = D^k \times (D^{n-k} - \{0\})$. By van Kampen's theorem, $\pi_1(X)$ is the pushout of the diagram

$$\begin{tikzcd} \pi_1(Y \cap N) \arrow[r] \arrow[d] & \pi_1(X) \\ \pi_1(N) & \end{tikzcd}$$

But $\pi_1(N) = 0$ and $\pi_1(Y \cap N) = 0$ if $k < n - 2$. Therefore $\pi_1(X) \to \pi_1(Y)$ is an isomorphism in this case. If $k = n - 2$, $\pi_1(Y \cap N) = \mathbf{Z}$, a generator being $\lambda = \{a\} \times S^{n-k-1}$ where $a \in e$ and $S^{n-k-1} = \partial D^{n-k}$. Thus λ is a small loop about e and $\pi_1(Y)$ is obtained from $\pi_1(X)$ by factoring out the normal subgroup represented by the image of λ.

COROLLARY 1.11. *If $\pi_1(M) = 0$, then $\pi_1(M - K)$ is generated normally by the x_i.*

2. DIOPHANTINE APPROXIMATION

The object of this section is to prove the following result.

THEOREM 2.1. *Let $K = Q(\sqrt{-m})$ be an imaginary quadratic number field, with ring of integers $\mathcal{O}$. Then there is a constant C depending on K with the following property. If z is any complex number not in K, there are an infinite number of solutions λ, μ of the inequality*

$$\left| z - \frac{\lambda}{\mu} \right| \leqslant \frac{C}{|\mu|^2}$$

with $\lambda, \mu \in \mathcal{O}$ and $(\lambda, \mu) = \mathcal{O}$.

Without the condition that λ and μ be relatively prime, this would be a standard result. If $\mathcal{O}$ has class number 1, the theorem as stated follows by reducing λ/μ to lowest terms. However, if the ideal (λ, μ) is not principal, a different approach is required.

LEMMA 2.2. *Theorem* 2.1 *holds if the condition* $(\lambda, \mu) = \mathcal{O}$ *is omitted.*

This is the known result mentioned above. A great deal of work has been done on determining the best value of the constant C for this result [14]. Of course, the value of C required for Theorem 2.1 will (presumably) be much greater than that required for Lemma 2.2. I will give a short proof of Lemma 2.2 here for convenience.

Proof. Let $1, \omega$ be a base for $\mathcal{O}$ as an Abelian group. Any complex number is congruent, mod $\mathcal{O}$, to a number in the parallelogram $P = \{x + y\omega \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$. Let M be a positive integer. Divide P into M^2 similar regions P_i by dividing the intervals $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ into M parts each. These small regions P_i have diameter dM^{-1} where d is the diameter of P. Let $\mu = r + s\omega$ where r, s are integers, $0 \leqslant r, s \leqslant M$. There are $(M + 1)^2$ values of μ so there are distinct μ_1, μ_2 with $\mu_1 z$, $\mu_2 z$ mod $\mathcal{O}$ lying in the same P_i. If $\mu = \mu_1 - \mu_2$, there is some $\lambda \in \mathcal{O}$ with $| \mu z - \lambda | \leqslant dM^{-1}$. Also $| \mu | \leqslant AM$ where $A = 1 + | \omega |$ so

$$\left| z - \frac{\lambda}{\mu} \right| < \frac{d}{M | \mu |} \leqslant \frac{B}{| \mu |^2}$$

with $B = Ad$. Since $d/M | \mu |$ can be made arbitrarily small by taking M large, and $z \notin K$, there will be an infinite number of λ, μ satisfying the inequality.

LEMMA 2.3. *Let* $K = Q(\sqrt{-m})$ *be an imaginary quadratic number field with ring of integers* $\mathcal{O}$. *Let* $\mathfrak{A}$ *be an ideal of* $\mathcal{O}$ *and let* $\gamma \in \mathfrak{A}$, $\gamma \neq 0$. *Then there is a constant* A *depending only on* $\mathfrak{A}$ *and* γ *so that if* $\mathfrak{A} = (\alpha, \beta)$ *with* $\beta \neq 0$, *there exist* $\lambda, \mu \in \mathcal{O}$ *with* $(\lambda, \mu) = \mathcal{O}$, $\mu\alpha - \lambda\beta = \gamma$, *and* $| \mu | \leqslant A | \beta |$.

Proof. We can clearly assume $\mathfrak{A} \subset \mathcal{O}$ by replacing $\mathfrak{A}$ by $\eta\mathfrak{A}$ for some $\eta \in \mathcal{O}$. Since $\mathcal{O}$ is a Dedekind ring, we can find $\delta \in \mathfrak{A}$ so that $\mathfrak{A} = (\gamma, \delta)$ [8, Cor. 18.22]. Since $\mathfrak{A}^{-1}\mathfrak{A} = \mathcal{O}$, we can find $\gamma', \delta' \in \mathfrak{A}^{-1}$ so $\gamma'\gamma + \delta'\delta = 1$. Choose fixed γ', δ', δ with these properties.

Now, let $\mathfrak{A} = (\alpha, \beta)$. Then there are $\alpha', \beta' \in \mathfrak{A}^{-1}$ such that

$\alpha'\alpha + \beta'\beta = 1$. Since $\mathfrak{A} \subset \mathcal{O} \subset \mathfrak{A}'$, we can replace α', β' by $\alpha' - \kappa\beta$, $\beta' + \kappa\alpha$ for any $\kappa \in \mathfrak{A}$. Choose κ so that $| \alpha'\beta^{-1} - \kappa | \leqslant d$, where d is a constant depending only on K (as in the proof of Lemma 1). Therefore we can assume that $| \alpha' | \leqslant d | \beta |$.

Let $A = \left(\begin{smallmatrix} \alpha' & -\beta \\ \beta' & \alpha \end{smallmatrix}\right)$ and $G = \left(\begin{smallmatrix} \gamma & \delta \\ -\delta' & \gamma' \end{smallmatrix}\right)$. Then $\det A = \det G = 1$. Clearly, AG has entries in $\mathcal{O}$ and $\det AG = 1$. Now $(\alpha, \beta)A = (1, 0)$ and $(1, 0)G = (\gamma, \delta)$ so $(\alpha, \beta)\, AG = (\gamma, \delta)$. Therefore, $\mu\alpha - \lambda\beta = \gamma$, where $AG = \left(\begin{smallmatrix} \mu & * \\ -\lambda & * \end{smallmatrix}\right)$. Thus $\mu = \alpha'\gamma - \beta\delta'$ so $| \mu | \leqslant | \gamma | \, | \alpha' | + | \delta' | \, | \beta | \leqslant (| \gamma | \, d + | \delta |) \, | \beta |$.

This proof was derived from that of Proposition 3.10 below, using explicit isomorphisms $\mathfrak{A} \oplus \mathfrak{A} \approx \mathcal{O} \oplus \mathcal{O}$.

Proof of Theorem 2.1. Let z be a complex number not in K. By Lemma 2.2 we can find $\alpha, \beta \in \mathcal{O}$ with

$$\left| z - \frac{\alpha}{\beta} \right| \leqslant \frac{B}{| \beta |^2}$$

and $/ \beta /$ arbitrarily large, B being the constant required in Lemma 2.2. Choose a set of representatives $\mathfrak{c}_0, \ldots, \mathfrak{c}_r$ for the ideal classes of $\mathcal{O}$. For each α, β, there is some i with $(\alpha, \beta) \sim \mathfrak{c}_i$ so there is some $u \in K$ with $u(\alpha, \beta) = \mathfrak{c}_i$. Let $N_0 = \max N\mathfrak{c}_i$, where N denotes the norm. Then $| u |^2 \leqslant N\mathfrak{c}_i(N(\alpha, \beta))^{-1} \leqslant N\mathfrak{c}_i \leqslant N_0$. Let $\alpha_1 = u\alpha$, $\beta_1 = u\beta$. Then

$$\left| z - \frac{\alpha_1}{\beta_1} \right| = \left| z - \frac{\alpha}{\beta} \right| \leqslant \frac{B}{| \beta |^2} = \frac{B}{| \beta_1 |^2} \, | u |^2 < \frac{B_1}{| \beta_1 |^2},$$

where $B_1 = N_0{}^2 B$. Since $| z - \alpha_1\beta_1^{-1} | = | z - \alpha\beta^{-1} |$ can be made arbitrarily small, we see that the inequality

$$\left| z - \frac{\alpha}{\beta} \right| \leqslant \frac{B}{| \beta |^2} \tag{1}$$

has an infinite number of solutions which satisfy the extra condition $(\alpha, \beta) = \mathfrak{c}_i$ for some i.

Let some $\gamma_i \in \mathfrak{c}_i$, $\gamma_i \neq 0$ for each i. Let A_i be the constant of Lemma 2.3 for γ_i, $\mathfrak{c}_i$ and let $A = \max A_i$. If α, β satisfy (1) and $(\alpha, \beta) = \mathfrak{c}_i$ for some i, we can, by Lemma 2.3, find $\mu, \lambda \in \mathcal{O}$, $(\mu, \lambda) = \mathcal{O}$ with $\mu\alpha - \lambda\beta = \gamma_i$ and $| \mu | \leqslant A | \beta |$. Let $G = \max | \gamma_i |$. If we choose $| \beta | > G$, then $\mu \neq 0$ (note $\lambda \in \mathcal{O}$ so $| \lambda | \geqslant 1$ if $\lambda \neq 0$, e.g., if $\mu = 0$), and we have

$$\left| z - \frac{\lambda}{\mu} \right| \leqslant \left| z - \frac{\alpha}{\beta} \right| + \left| \frac{\alpha}{\beta} - \frac{\lambda}{\mu} \right| \leqslant \frac{B_1}{| \beta |^2} + \left| \frac{\gamma_i}{\beta\mu} \right| \leqslant \frac{BA^2}{| \mu |^2} + \frac{GA}{| \mu |^2} = \frac{C}{| \mu |^2}.$$

Also $\mu \in \mathcal{O}$ so $|\mu| \geqslant 1$. Therefore

$$\left| z - \frac{\lambda}{\mu} \right| \leqslant \frac{B_1}{|\beta|^2} + \frac{B}{|\beta|}.$$

Since this can be made arbitrarily small by choosing $|\beta|$ large, an infinite number of λ/μ must occur.

3. Binary Hermitian Forms

Let P be the set of all positive definite Hermitian forms in 2 complex variables. Thus P is the set of all functions f on $\mathbf{C}^2$ of the form $f(u, v) = A \mid u \mid^2 + 2\mathscr{R}(B\bar{u}v) + C \mid v \mid^2$ where A, C are real, and $f(u, u) > 0$ for all $u \neq 0$. By completing the square, we may write $f(u, v) = A[\mid u + zv \mid^2 + \zeta^2 \mid v \mid^2]$, where $A > 0$, $z \in \mathbf{C}$, $\zeta > 0$. Clearly, A, z and ζ are uniquely determined by f. Since we are mainly interested in z and ζ, we define an equivalence $f \sim g$ if there is a constant $a > 0$ such that $f(p) = ag(p)$ for all $p \in \mathbf{C}^2$. Let H be the set of equivalence classes. Then H may be identified with the space of pairs (z, ζ) with $z \in \mathbf{C}$, $\zeta > 0$, i.e., the upper half space of $\mathbf{C} \times \mathbf{R}$.

The group $GL(2, \mathbf{C})$ is the group of all linear automorphisms of $\mathbf{C}^2$ and hence acts on P and on H by $\sigma f \cdot (p) = f(\sigma^{-1}p)$ for $\sigma \in GL(2, \mathbf{C})$ $p \in \mathbf{C}^2$. The action of σ in terms of z and ζ is easily computed [2].

Lemma 3.1. *If* $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbf{C})$, *the action of* σ *on* H *is given by* $\sigma(z, \zeta) = (z', \zeta')$ *where*

$$\zeta' = \frac{|\Delta| \zeta}{|cz - d|^2 + \zeta^2 |c|^2}$$

$$z' = \frac{(\bar{d} - \bar{c}\bar{z})(az - b) - \zeta^2 \bar{c} a}{|cz - d|^2 + \zeta^2 |c|^2}$$

with $\Delta = \det \sigma = ad - bc$.

Proof. We have

$$\sigma^{-1} = \Delta^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \text{so} \quad \sigma f \cdot (u, v) = \Delta^{-2} f(du - bv, -cu + dv).$$

We must reduce this to canonical form by completing the square. There is no need to reproduce the details here.

COROLLARY 3.2.

(1) *If* $\sigma = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, *then* $\sigma(z, \zeta) = (z - s, \zeta)$,

(2) *If* $\sigma = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, *then* $\sigma(z, \zeta) = (d^{-1}az, | d^{-1}a | \zeta)$,

(3) *If* $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, *then* $\sigma(z, \zeta) = \left(\frac{\bar{z}}{| z |^2 + \zeta^2}, \frac{\zeta}{| z |^2 + \zeta^2}\right)$.

We now show that $GL(2, \mathbf{C})$ acts on H as a group of isometries [2].

LEMMA 3.3. *The riemannian metric* $ds^2 = \zeta^{-2}(d\xi^2 + d\eta^2 + d\zeta^2)$ *on* H, *where* $z = \xi + i\eta$, *is invariant under the action of* $GL(2, \mathbf{C})$.

Proof. Since $GL(2, \mathbf{C})$ is generated by the matrices considered in Corollary 3.2, it is sufficient to check the invariance under these transformations. Again I will omit the details.

LEMMA 3.4. (1) *The set consisting of all hemispheres in H with center in the plane $\zeta = 0$, together with all vertical (half) planes in H, is stable under the action of $GL(2, \mathbf{C})$*

(2) *The set consisting of all vertical semicircles in H with center in the plane $\zeta = 0$, together with all vertical (half) lines in H, is stable under the action of $GL(2, \mathbf{C})$.*

(3) *The set considered in* (2) *consists exactly of all geodesics for the metric defined in Lemma* 3.3.

Proof. The set considered in (1) consists exactly of those subsets of H which can be defined by an equation of the form $f(a, b) = f(c, d)$ where f is the Hermitian form corresponding to (z, ζ) and $a, b, c, d \in \mathbf{C}$. The vertical planes correspond to the case $| b | = | d |$. These equations are clearly preserved by the action of $GL(2, \mathbf{C})$ except for the values of a, b, c, d.

The set considered in (2) consists exactly of the nontrivial intersections of pairs of elements of the set considered in (1).

Now it is clear that each vertical line L in H is a geodesic because if $p, q \in L$ are joined by a curve C, any horizontal component of C will only serve to increase its length. If $p, q \in H$ do not lie on a vertical line, there is a unique semicircle C through p and q with center in the plane $\zeta = 0$. Let $(z_0, 0)$ be one of the ends of C. Let $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & -z_0 \end{pmatrix} \in GL(2, \mathbf{C})$

and $\sigma(z, \zeta) = (z', \zeta')$. If $(z, \zeta) \in C$ approaches $(z_0, 0)$, we have $|z - z_0| = O(\zeta^2)$ so

$$\zeta' = \frac{\zeta}{|z - z_0|^2 + \zeta^2} = O\left(\frac{1}{\zeta}\right) \to \infty.$$

Therefore σC is not a semicircle and so by (2), can only be a vertical line. Thus it is a geodesic and, since σ is an isometry, C is also a geodesic. Therefore, the curves considered in (2) are all geodesics. If $p, q \in C$ as above, the segment of σC between σp and σq is the unique shortest line between σp and σq and is even the unique line between σp and σq for which the length function is stationary. Since σ is an isometry, the segment of C between p and q has the same property. Thus we have found all geodesics.

Now let $K = Q(\sqrt{-m})$ be an imaginary quadratic number field with ring of integers $\mathcal{O}$. Consider the action of the group $SL(2, \mathcal{O}) \subset GL(2, \mathbf{C})$ on H. The Bianchi–Humbert theory gives a fundamental domain for this action. Consider the lattice $\mathcal{O}^2 = \mathcal{O} \times \mathcal{O} \subset \mathbf{C}^2$. Consider all points $(\mu, \lambda) \in \mathcal{O}^2$ such that the ideal $(\mu, \lambda) = \mathcal{O}$. If f is a binary Hermitian form, look at the values $f(\mu, \lambda)$ for points of this type. Among these, there is a minimum value called the proper minimum of f [10]. It may of course be larger than the true minimum of f over all points of $\mathcal{O}^2$ other than (0, 0). If μ and λ generate the unit ideal there is an element $\sigma \in SL(2, \mathcal{O})$ such that $(1, 0) = \sigma(\mu, \lambda)$. Therefore we can find $\sigma \in SL(2, \mathcal{O})$ such that σf takes its proper minimum at (1, 0).

Definition. If $\mu, \lambda \in \mathcal{O}$ generate the unit ideal and $\mu \neq 0$, let $S_{\mu,\lambda}$ denote the hemisphere in H given by $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 = 1$. This has center $(\lambda/\mu, 0)$ and radius $1/|\mu|$. Let B be the set of points in H which lie above or on all $S_{\mu\lambda}$, i.e., B is the set of $(z, \zeta) \in H$ satisfying the inequalities $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 \geqslant 1$ for all $\mu, \lambda \in \mathcal{O}$ which generate the unit ideal.

Lemma 3.5. *Every point of H can be transformed into a point of B by some $\sigma \in SL(2, \mathcal{O})$.*

Proof. The point (z, ζ) corresponds to the form $f(u, v) = |u + zv|^2 + \zeta^2 |v|^2$. The definition of B asserts $f(-\lambda, \mu) \geqslant f(1, 0)$ for all $(\lambda, \mu) = \mathcal{O}$. In other words $(z, \zeta) \in B$ if and only if f takes its proper minimum at (1, 0).

From now on, I will work directly with the coordinates (z, ζ) and ignore the form to which they correspond. An alternative characterization of the set B will be very useful.

LEMMA 3.6. *A point* (z, ζ) *of* H *lies in* B *if and only if for every* $\sigma \in SL(2, \mathcal{O})$ *we have* $\sigma(z, \zeta) = (z', \zeta')$ *with* $\zeta' \leqslant \zeta$.

In other words, to transform a point p of H into B we have only to find that transform of p whose ζ-coordinate is largest.

Proof. Since $\det \sigma = 1$, Lemma 3.1 shows that $\zeta' \leqslant \zeta$ if and only if $| cz - d |^2 + | c |^2\zeta^2 \geqslant 1$. The c, d which can occur as the second row of some $\sigma \in SL(2, \mathcal{O})$ are exactly the c, d with $(c, d) = \mathcal{O}$.

LEMMA 3.7. *Let* $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \mu & \lambda \end{smallmatrix}\right) \in SL(2, \mathcal{O})$ *with* $\mu \neq 0$. *Then* $B \cap \sigma^{-1}B = B \cap S_{\mu,\lambda}$.

Proof. Let $(z, \zeta) \in B$, $\sigma(z, \zeta) = (z', \zeta')$. Then by Lemma 3.6, $(z', \zeta') \in B$ if and only if ζ' is maximal among all transforms of the point, i.e., if and only if $\zeta' = \zeta$. Since $\det \sigma = 1$, Lemma 3.1 shows this happens if and only if $| \mu z - \lambda |^2 + | \mu |^2\zeta^2 = 1$, i.e., if and only if $(z, \zeta) \in S_{\mu,\lambda}$.

LEMMA 3.8. *Let* Φ *be the subgroup of* $SL(2, \mathcal{O})$ *consisting of all matrices of the form* $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$. *Then* $\sigma B = B$ *if and only if* $\sigma \in \Phi$. *If* $\sigma \notin \Phi$, *then* $\operatorname{int} B \cap \sigma \operatorname{int} B = \emptyset$.

Proof. If $T_s = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)$ with $s \in \mathcal{O}$, then $T_s S_{\mu,\lambda} = S_{\mu,\lambda+s\mu}$. If ρ is a unit of $\mathcal{O}$ and $L_\rho = \left(\begin{smallmatrix} \rho^{-1} & 0 \\ 0 & \rho \end{smallmatrix}\right)$, then $L_\rho S_{\mu,\lambda} = S_{\mu,\rho^2\lambda}$. Therefore Φ merely permutes the equations defining B. If $\sigma \notin \Phi$, then $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \mu & \lambda \end{smallmatrix}\right)$ and $\sigma B \cap B = B \cap S_{\mu\lambda}$ by Lemma 3.7. This lies on the boundary of B since no point under $S_{\mu\lambda}$ is in B.

Now choose a fundamental domain F for the group of translations of by elements of $\mathcal{O}$. For example, if $1, \omega$ is a base for $\mathcal{O}$ as an Abelian group and $\omega = \omega_1 + i\omega_2$ we may let F be the set of all $z = \xi + i\eta$ with $-\omega_2/2 \leqslant \eta \leqslant \omega_2/2$ and $-1/2 \leqslant \xi \leqslant 1/2$.

DEFINITION. Let D be the set of all $(z, \zeta) \in B$ with $z \in F$. Given any point of H we can transform it into B by an element of $SL(2, \mathcal{O})$ and then into D by some $T_s = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)$. Thus $SL(2, \mathcal{O})D = H$.

Remark. If $K \neq Q(\sqrt{-1})$ or $Q(\sqrt{-3})$, the only units of $\mathcal{O}$ are ± 1 and it follows from Lemma 3.8 that $\sigma \operatorname{int} D \cap \operatorname{int} D = \emptyset$ except for $\sigma = 1$ and $\sigma = J = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Note J acts trivially on H. Thus D is a fundamental domain in the usual sense for $PSL(2, \mathcal{O})$ for $K \neq Q(\sqrt{-1})$ or $Q(\sqrt{-3})$.

PROPOSITION 3.9. *The set D is $SL(2, \mathcal{O})$-normal,* i.e., *each point of H has a neighborhood meeting σD for only a finite number of $\sigma \in SL(2, \mathcal{O})$.*

Proof. Let $(z_0, \zeta_0) \in H$. Choose ϵ with $0 < \epsilon < \zeta_0$. Let U be a small circular neighborhood of z_0. Let W be the set of (z, ζ) with $z \in U$, $\zeta > \epsilon$. Then W is a neighborhood of (z_0, ζ_0). Suppose $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathcal{O})$ and $W \cap \sigma D \neq \emptyset$. Then there is some $(z, \zeta) \in D$ with $\sigma(z, \zeta) = (z', \zeta')$, $z' \in U$ and $\zeta' > \epsilon$. By Lemma 3.1,

$$\zeta' = \frac{\zeta}{|\, cz - d \,|^2 + \zeta^2 \,|\, c \,|^2} \leqslant \frac{\zeta}{\zeta^2 \,|\, c \,|^2} = \frac{1}{\zeta \,|\, c \,|^2}$$

so $|\, c \,|^2 \leqslant \zeta^{-1}\zeta'^{-1}$. Since $(z, \zeta) \in D \subset B$, $\zeta \geqslant \zeta' > \epsilon$, so $|\, c \,|^2 \leqslant \epsilon^{-2}$. Therefore there are only a finite number of possibilities for c.

Now for fixed c, we have $\epsilon < \zeta' \leqslant \zeta \,|\, cz - d \,|^{-2}$. But $1 \geqslant \zeta\zeta' \,|\, c \,|^2 \geqslant \zeta\epsilon \,|\, c \,|^2$ so $\zeta \leqslant \epsilon^{-1} \,|\, c \,|^{-2}$. Therefore $|\, cz - d \,|^2 \leqslant \epsilon^{-1}\zeta \leqslant \epsilon^{-2} \,|\, c \,|^{-2}$ so $|\, d \,| \leqslant |\, cz - d \,| + |\, cz \,| \leqslant \epsilon^{-1} \,|\, c \,|^{-1} + |\, cz \,|$. But $z \in F$ so $|\, z \,|$ is bounded. Thus there are only a finite number of possibilities for d.

Finally if $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\sigma' = \left(\begin{smallmatrix} a' & b' \\ c & d \end{smallmatrix}\right)$ are in $SL(2, \mathcal{O})$, then there is some $s \in \mathcal{O}$ so that $\sigma' = T_s\sigma$, where $T_s = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)$. If $\sigma(z, \zeta) = (z', \zeta') \in W$, then $\sigma'(z, \zeta) = T_s(z', \zeta') = (z' - s, \zeta')$. Since U is bounded, there are only a finite number of s such that $T_s W \cap W \neq \emptyset$.

We must show now that D meets only a finite number of the $S_{\mu,\lambda}$ [10].

PROPOSITION 3.10. *Let R be a Dedekind ring and M a torsion free finitely generated R-module of rank r. Let $a_1, \ldots, a_s$ and $b_1, \ldots, b_s$ be generating sets for M with $s > r$. Then there is some $(c_{ij}) \in SL(s, R)$ with $b_i = \sum c_{ij} a_j$.*

The exceptional case $s = r$ can only occur when M is free on r generators. In this case any element of $GL(r, R)$ can obviously occur.

Proof. Let F be free on s generators. Then (a_i), (b_i) give maps $a, b : F \to M$. Consider the short exact sequences

$$0 \longrightarrow A \longrightarrow F \xrightarrow{a} M \longrightarrow 0,$$

$$0 \longrightarrow B \longrightarrow F \xrightarrow{b} M \longrightarrow 0.$$

Since M is projective [5] these split and $A \oplus M \approx F \approx B \oplus M$. By [12] we may write $A = F' \oplus \mathfrak{a}$, $B = F'' \oplus \mathfrak{b}$ where F', F'' are free, and $\mathfrak{a}$, $\mathfrak{b}$ are nonzero ideals of R. Since A and B have the same rank there is an isomorphism $\psi : F' \approx F''$. By [12], $A \oplus M \approx B \oplus M$ implies that there is an isomorphism $\varphi : \mathfrak{a} \approx \mathfrak{b}$. Let s be a unit of R. Define $\theta_s : F \to F$ to be the isomorphism obtained by composing the isomorphisms $F \approx M \oplus F' \oplus \mathfrak{a}$ and $F \approx M \oplus F'' \oplus \mathfrak{b}$ with the isomorphism $(1, \psi, s\varphi) : M \oplus F' \oplus \mathfrak{a} \to M \oplus F'' \oplus \mathfrak{b}$. The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & F & \xrightarrow{a} & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \theta_s} & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & B & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

commutes so we may let (c_{ij}) be the matrix representing θ_s^{-1}. Clearly $\det \theta_s = s \det \theta_1$ so we may choose s so that $\det \theta_s = 1$.

Remark. As a consequence of this we can easily obtain a classical result of Bianchi and Hurwitz [11]. Let R be a Dedekind ring with quotient field K and ideal class group C. Let $P^n(K)$ be the n-dimensional projective space over K, $n \geqslant 1$. Any point $p \in P^n(K)$ has homogeneous coordinates $(\alpha_0, \ldots, \alpha_n)$ with $\alpha_i \in R$. Define a function $i : P^n(K) \to C$ by letting $i(\alpha_0, \ldots, \alpha_n)$ be the class of the ideal generated by $\alpha_0, \ldots, \alpha_n$. It is easy to see that i is well-defined, onto, and that $i(p) = i(\sigma p)$ for $\sigma \in GL(n+1, R)$. Now, if $i(p) = i(q)$, we may choose homogeneous coordinates $p = (\alpha_0, \ldots, \alpha_n)$ $q = (\beta_0, \ldots, \beta_n)$ which generate the same ideal of R. By Proposition 3.10, we can find $\sigma \in SL(n+1, R)$ so that $\sigma p = q$. Therefore the maps

$$P^n(K)/SL(n+1, R) \to P^n(K)/GL(n+1, R) \to C$$

are one to one and onto.

We now return to the case of a quadratic imaginary field $K = Q(\sqrt{-m})$.

PROPOSITION 3.11. *Let $K = Q(\sqrt{-m})$ be a quadratic imaginary field with ring of integers $\mathcal{O}$. If $z \in \mathbf{C}$, we can find $\mu, \lambda \in \mathcal{O}$ with $(\mu, \lambda) = \mathcal{O}$ and $| \mu z - \lambda | \leqslant 1$. The "singular points" for which we cannot find μ, λ with $(\mu, \lambda) = \mathcal{O}$, $| \mu z - \lambda | < 1$ all lie in K. There are a finite number of points $\alpha_1, \ldots, \alpha_r \in K$ such that the singular points are exactly the points $\alpha_\nu + \gamma$ for $\nu = 1, \ldots, r$, $\gamma \in \mathcal{O}$.*

Proof. If $z \notin K$, this is a consequence of Theorem 2.1. Suppose now $z \in K$. Write $z = \alpha/\beta$ with $\alpha, \beta \in \mathcal{O}$. Let $\mu = \beta^2$, $\lambda = 1 + \alpha\beta$. Then $(\mu, \lambda) = \mathcal{O}$ since

$$\lambda(1 - \alpha\beta) + \alpha^2\mu = 1 \qquad \text{and} \qquad |\mu z - \lambda| = |\beta\alpha - 1 - \alpha\beta| = 1.$$

Now let $\mathfrak{A}$ be the ideal (α, β). If $\mathfrak{A} = (\gamma, \delta)$, Proposition 3.10 shows that there are $\mu, \lambda \in \mathcal{O}$ with $(\lambda, \mu) = \mathcal{O}$ and $\mu\alpha - \lambda\beta = \gamma$. Therefore $|\mu z - \lambda| = |\gamma|/|\beta|$. Therefore z is not singular if $|\gamma| < |\beta|$. Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_h$ represent all ideal classes. For each i, let γ_{ij} be the nonzero elements of $\mathfrak{A}_i$ with minimal $|\gamma_{ij}|$. If $z = \alpha/\beta$, we can assume $\mathfrak{A} = (\alpha, \beta) = \mathfrak{A}_i$ for some i by replacing α, β by $u\alpha, u\beta$ for some $u \in K$. If $\beta \neq \gamma_{ij}$ for any j, then α/β is nonsingular. If $\beta = \gamma_{ij}$, there are only a finite number of possibilities for β. Given β, $\mathfrak{A}/(\beta)$ is finite so there are only a finite number of possibilities for α modulo β. If $\gamma \in \mathcal{O}$, it is clear that z is singular if and only if $z + \gamma$ is. The last statement of Proposition 3.11 clearly follows from these results.

A more detailed discussion of the singular points is given in Section 7.

LEMMA 3.12. (1) *If $s \in \mathbf{C}$ is not a singular point, there is a neighborhood U of s and an $\epsilon > 0$ so that every $(z, \zeta) \in B$ with $z \in U$ has $\zeta > \epsilon$.*

(2) *If $s \in \mathbf{C}$ is a singular point, there is a neighborhood U of s and an $\epsilon > 0$ so that every $(z, \zeta) \in B$ with $z \in U$ has $\zeta > \epsilon\, |z - s|^{1/2}$.*

Proof. For (1), choose $\mu, \lambda \in \mathcal{O}$ so $|\mu s - \lambda| < 1$, $(\mu, \lambda) = \mathcal{O}$. Let U be a neighborhood of s and let $0 < \theta < 1$ so that for $z \in U$ we have $|\mu z - \lambda| < \theta$. Since all points of B satisfy $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 \geqslant 1$ we can take $\epsilon = |\mu|^{-1}(1 - \theta^2)^{1/2}$.

Suppose now $s = \alpha/\beta$ is singular. Let $1, \omega$ be a base for $\mathcal{O}$. Then we have $|\mu s - \lambda| = 1$ and $(\mu, \lambda) = \mathcal{O}$ for $\mu = \beta^2$, $\lambda = \alpha\beta \pm 1$ and also for $\mu = \omega\beta^2$, $\lambda = \omega\alpha\beta \pm 1$.

The two circles $|\beta^2 z - \alpha\beta \pm 1| = 1$ pass through $z = s$ and are tangent there along the line $\mathscr{R}(\beta^2 z - \alpha\beta) = 0$. The two circles $|\omega\beta^2 z - \alpha\beta \pm 1| = 1$ also pass through $z = s$ and are tangent along the line $\mathscr{R}(\omega\beta^2 - \omega\alpha\beta) = 0$. Since these lines are distinct, each of the first pair of circles overlaps each of the second pair. Therefore a neighborhood of s may be divided into 4 parts each lying inside one of the circles and bounded by two secants passing through s.

For each of these circles, choose coordinates in the z plane so that the circle has the equation $(x - r)^2 + y^2 = r^2$, the point s having the coordinates $(0, 0)$. The secants will have equation $y = m_1 x$ and $y = m_2 x$. The equation

$$|\mu z - \lambda|^2 + |\mu|^2 \zeta^2 \geqslant 1 \qquad \text{or} \qquad |z - \lambda/\mu|^2 + \zeta^2 \geqslant |\mu|^{-2}$$

now becomes $(x - r)^2 + y^2 + \zeta^2 \geqslant r^2$ or $\zeta^2 \geqslant 2rx - x^2 - y^2$. Between the secants, $|y| \leqslant M|x|$ where $M = \max(|m_1|, |m_2|)$ so $\zeta^2 \geqslant 2rx - x^2 - M^2x^2 > \epsilon^2 x$ if $2r > \epsilon^2$ and x is close enough to 0.

THEOREM 3.13. *There are only a finite number of $\lambda, \mu \in \mathcal{O}$ with $(\lambda, \mu) = \mathcal{O}$ such that $D \cap S_{\mu,\lambda} \neq \emptyset$.*

Proof. For a given μ, $D \cap S_{\mu,\lambda} = \emptyset$ unless the distance from λ/μ to F is $\leqslant |\mu|^{-1}$. Therefore if d is the diameter of F, $D \cap S_{\mu,\lambda} = \emptyset$ unless $|\lambda/\mu| \leqslant d + |\mu|^{-1}$. There are only a finite number of λ satisfying this. Thus it will suffice to find a bound for μ.

Let $s_1, \ldots, s_n$ be the singular points in F. Let U_i, ϵ_i satisfy (2) of Lemma 3.12 for s_i. Let $\epsilon = \min \epsilon_i$. Replace each U_i with a smaller open neighborhood V_i such that $z \in V_i$ implies $|z - s_i| < \epsilon^2/2$. Since $F - \bigcup V_i$ is compact, Lemma 3.12 (1) implies that there is an $\eta > 0$ such that for $(z, \zeta) \in D$, $z \notin \bigcup V_i$, we have $\zeta > \eta$. If $|\mu| > \eta^{-1}$, the highest point on $S_{\lambda,\mu}$ has $\zeta = |\mu|^{-1} < \eta$. Thus $S_{\mu,\lambda}$ can only meet D at points (z, ζ) with $z \in \bigcup V_i$. Suppose $z \in V_i$. Then $\zeta > \epsilon |z - s_i|^{1/2}$. But $(z, \zeta) \in S_{\mu,\lambda}$ so $|z - \lambda\mu^{-1}|^2 + \zeta^2 = |\mu|^{-2}$. Thus $|z - \lambda\mu^{-1}|^2 + \epsilon^2 |z - s| < |\mu|^{-2}$, where we have written $s = s_i$. Let $u = z - s$. Since s is singular $|\mu s - \lambda| \geqslant 1$ or $|s - \mu^{-1}\lambda| \geqslant |\mu|^{-1}$. Let $t = s - \mu^{-1}\lambda$ so $|t| \geqslant |\mu|^{-1}$. Then $z - \mu^{-1}\lambda = u + t$ so $|u + t|^2 + \epsilon^2 |u| < |\mu|^{-2}$. In particular, $|u + t| < |\mu|^{-1}$. Now $|\mu|^{-1} \leqslant |t| \leqslant |u + t| + |u|$ so

$$\begin{aligned} |\mu|^{-2} &= |u + t|^2 + 2|u|\,|u + t| + |u|^2 \\ &< |\mu|^{-2} - \epsilon^2 |u| + 2|\mu|^{-1}|u| + |u|^2 \end{aligned}$$

or

$$0 < -\epsilon^2 |\mu| + 2|\mu|^{-1}|u| + |u|^2$$

so $u \neq 0$ and $0 < -\epsilon^2 + 2|\mu|^{-1} + |u|$. Now $|u| < \epsilon^2/2$ by the choice of V_i so $0 < -\epsilon^2/2 + 2|\mu|^{-1}$. Therefore $|\mu| \leqslant 4\epsilon^{-2}$.

COROLLARY 3.14. *There are only a finite number of $\sigma \in SL(2, \mathcal{O})$ such that $\sigma D \cap D \neq \emptyset$.*

Proof. Let $\sigma^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathcal{O})$. Since $\sigma D \cap D \subset \sigma B \cap B = B \cap S_{c,d}$ by Lemma 3.7 we have $\sigma D \cap D \subset D \cap S_{c,d}$. If $\sigma D \cap D \neq \emptyset$, Theorem 3.13 shows that there are only a finite number of possibilities for c and d. If $\sigma'^{-1} = \left(\begin{smallmatrix} a' & b' \\ c & d \end{smallmatrix}\right) \in SL(2, \mathcal{O})$, then $\sigma'^{-1} = T_s \sigma^{-1}$ for $T_s = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)$, $s \in \mathcal{O}$. To show there are only a finite number of s with $T_s \sigma^{-1} D \cap D \neq \emptyset$, it will suffice to show that $|z'|$ is bounded for $(z', \zeta') \in \sigma D$. If $(z, \zeta) \in D$, $\sigma(z, \zeta) = (z', \zeta')$ with

$$z' = \frac{(\bar{d} - \bar{c}\bar{z})(az - b) - \zeta^2 \bar{c} a}{|cz - d|^2 + \zeta^2 |c|^2}.$$

Since $(z, \zeta) \in D$, the denominator is $\geqslant 1$. Therefore $|z'|$ is bounded for bounded ζ. But as $\zeta \to \infty$, $z' \to -ac^{-1}$ uniformly for $z \in F$.

4. A Presentation for $SL(2, \mathcal{O})$

Let $K = Q(\sqrt{-m})$ be an imaginary quadratic number field with ring of integers $\mathcal{O}$. Let H and B be as in Section 3. Let $\Phi \subset SL(2, \mathcal{O})$ be the subgroup of $SL(2, \mathcal{O})$ considered in Lemma 3.8.

Lemma 4.1. *If $p \in H$, the set of $\sigma \in SL(2, \mathcal{O})$ such that $\sigma p \in B$ is a finite union of right cosets $\Phi \sigma_\nu$.*

Proof. It is clearly a union of right cosets by Lemma 3.8. Now, if $\sigma p \in B$, there is some $s \in \mathcal{O}$ with $T_s \sigma p \in D$. By Proposition 3.9, there are only a finite number of possibilities for $T_s \sigma$.

Definition. Let $d(p)$ be one less than the number of right cosets $\Phi \sigma_\nu$ occurring in Lemma 4.1.

This function plays a crucial role in the arguments of this section. Clearly $d(\sigma p) = d(p)$ for $\sigma \in SL(2, \mathcal{O})$. We now determine $d(p)$ for $p \in B$. We first observe that if $(\mu, \lambda) = \mathcal{O}$, $(\beta, \alpha) = \mathcal{O}$, and $\lambda/\mu = \alpha/\beta$, then $\alpha = u\lambda$, $\beta = u\mu$ for some unit $u \in \mathcal{O}$. This in turn implies that $S_{\beta,\alpha} = S_{\mu,\lambda}$. Conversely, $S_{\mu,\lambda}$ determines the number λ/μ since $(\lambda/\mu, 0)$ is the center of $S_{\mu,\lambda}$. Thus we can specify the surface $S_{\mu,\lambda}$ unambiguously by giving the number λ/μ.

Definition. Let P be the set of $\alpha \in K$ of the form $\alpha = \lambda/\mu$ with $\lambda, \mu \in \mathcal{O}$, $(\lambda, \mu) = \mathcal{O}$. If $\alpha \in P$, define $S(\alpha) = S_{\mu,\lambda}$ for any such λ, μ.

LEMMA 4.2. *Let $p \in B$. Then $d(p)$ is the number of $\alpha \in P$ such that $p \in S(\alpha)$.*

Proof. Since $p \in B$, we see that $\sigma p \in B$ if and only if $p \in B \cap \sigma^{-1}B$. Let $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \mu & \lambda \end{smallmatrix}\right)$. If $\mu = 0$, $\sigma^{-1}B = B$ by Lemma 3.8. These σ form the coset Φ. If $\mu \neq 0$, Lemma 3.7 shows that $\sigma p \in B$ if and only if $p \in S_{\mu,\lambda}$ We must show that the coset $\Phi\sigma$ determines λ/μ and conversely.

LEMMA 4.3. *Let*

$$\sigma = \begin{pmatrix} \lambda_1 & \mu_1 \\ \mu & \lambda \end{pmatrix}, \qquad \tau = \begin{pmatrix} \alpha_1 & \beta_1 \\ \beta & \alpha \end{pmatrix} \in SL(2, \mathcal{O}),$$

with $\mu \neq 0$, $\beta \neq 0$. Then

(1) *$\Phi\sigma = \Phi\tau$ if and only if $\alpha/\beta = \lambda/\mu$,*

(2) *$\Phi\sigma\Phi = \Phi\tau\Phi$ if and only if there is an element $\gamma \in \mathcal{O}$ and a unit $u \in \mathcal{O}$ such that $\alpha/\beta = u^2(\lambda/\mu) + \gamma$.*

Part (1) finishes the proof of Lemma 4.2. Part (2) will be needed later.

Proof. If $\varphi = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Phi$, then $\varphi\sigma = \left(\begin{smallmatrix} * & * \\ d\mu & d\lambda \end{smallmatrix}\right)$ so $\Phi\sigma = \Phi\tau$ implies $\alpha/\beta = \lambda/\mu$. Conversely $\lambda/\mu = \alpha/\beta$ implies $\alpha = d\lambda$, $\beta = d\mu$ with $d \in \mathcal{O}$. But $\sigma, \tau \in SL(2, \mathcal{O})$ so d must be a unit of $\mathcal{O}$. Let $a = d^{-1}$ and $\varphi = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$. Then $\varphi\sigma = \left(\begin{smallmatrix} * & * \\ \beta & \alpha \end{smallmatrix}\right)$ so $\varphi\sigma\tau^{-1} = \left(\begin{smallmatrix} * & * \\ \beta & \alpha \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha & -\beta_1 \\ -\beta & \alpha_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in \Phi$ and $\Phi\sigma = \Phi\tau$.

For (2), let $\varphi = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ and $\psi = \left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right)$. If

$$\tau = \varphi\sigma\psi = \begin{pmatrix} * & * \\ d\mu a' & d\mu b' + d\lambda d' \end{pmatrix} \qquad \text{then} \quad \frac{\alpha}{\beta} = \frac{d\mu b' + d\lambda d'}{d\mu a'} = \frac{d'}{a'}\frac{\lambda}{\mu} + \frac{b'}{a'}.$$

But $a'd' = 1$. Let $u = d'$, $\gamma = b'/a'$. Conversely, if $\alpha/\beta = u^2(\lambda/\mu) + \gamma$, choose $\psi = \left(\begin{smallmatrix} u^{-1} & u^{-1}\gamma \\ 0 & u \end{smallmatrix}\right)$. By (1), $\Phi\sigma\psi = \Phi\tau$.

We now look more closely at the structure of B. If F is any bounded set of $\mathbf{C}$, it follows from Theorem 3.13 that the part of B over F, i.e., the set of $(z, \zeta) \in B$ with $z \in F$, is bounded below by a finite number of $S_{\mu,\lambda}$. Therefore the part of B over F is defined by $z \in F$ and a finite number of inequalities $|\mu z - \lambda|^2 + |\mu|^2 \zeta^2 \geqslant 1$. It follows that the interior of B is defined by the inequalities $|\mu z - \lambda|^2 + |\mu|^2 \zeta^2 > 1$ for all $(\mu, \lambda) = \mathcal{O}$ and that the boundary ∂B consists of those points of B which lie on some $S_{\mu,\lambda}$. Also, if $(z, \zeta) \in B$ then $(z, \zeta') \in \operatorname{int} B$ for all $\zeta' > \zeta$ Since all $S_{\mu,\lambda}$ lie in the set where $\zeta \leqslant 1$, we see that the set of all (z, ζ) with $\zeta > 1$ lies in int B. Since every point of B can be joined to this set by a vertical line we see that B and int B are pathwise connected.

It is also clear that vertical projection maps ∂B homeomorphically onto the plane $\zeta = 0$.

Consider now two hemispheres $S_{\mu,\lambda}$ and $S_{\beta,\alpha}$. Let $P_{\mu,\lambda;\beta,\alpha}$ be the vertical plane defined by the equation

$$\left| z - \frac{\lambda}{\mu} \right|^2 + \zeta^2 - \frac{1}{|\mu|^2} = \left| z - \frac{\alpha}{\beta} \right|^2 + \zeta^2 - \frac{1}{|\beta|^2}.$$

On one side of this plane, $S_{\mu,\lambda}$ lies strictly below $S_{\beta,\alpha}$. On the other side, $S_{\beta,\alpha}$ lies strictly below $S_{\mu,\lambda}$. On the plane itself, $S_{\mu,\lambda}$ and $S_{\beta,\alpha}$ meet in a geodesic semicircle (if they meet at all). Let $e_{\mu,\lambda} = B \cap S_{\mu,\lambda}$. The above argument shows that it is the intersection of $S_{\mu,\lambda}$ with a finite number of vertical half spaces. Therefore $e_{\mu,\lambda}$ is either empty, a point, an arc of a geodesic, or a 2-cell bounded by a finite number of geodesic arcs. Also, the projection of $e_{\mu,\lambda}$ on $\mathbf{C}$ is a convex polyhedral set. The $e_{\mu,\lambda}$ which are 2-cells determine a regular cell subdivision of ∂B whose projection on $\mathbf{C}$ is a convex polyhedral cell subdivision of $\mathbf{C}$. It is clearly invariant under the group Φ. The terms "vertex", "edge", and "2-cell" will refer to this cell subdivision. The term "open" will be used in the usual combinatorial sense. For example, an open geodesic segment will consist of all points on a geodesic between two points p and q, the points p and q themselves being excluded. An open vertex is just the vertex itself.

LEMMA 4.4. *Let S be a hemisphere in H with center in the plane $\zeta = 0$. Let e be an open geodesic segment of H which meets S but does not pierce S. Then $e \subset S$.*

Proof. This is clear if e is a segment of a vertical line. Suppose e is a segment of a vertical semicircle with center in the plane $\zeta = 0$. Let P be the vertical plane containing e. In P, e lies on a circle C and $P \cap S$ lies on a circle C'. If C and C' do not meet we are done. If C and C' meet at 2 points, neither of these points can lie on e by the hypothesis. If C and C' are tangent, their point of contact lies on the line joining their centers and hence in the plane $\zeta = 0$. Therefore it does not lie on e and again we are done. The only remaining case is the case $C' = C$.

COROLLARY 4.5. *If e is an open 2-cell of ∂B such that $\bar{e} = B \cap S_{\mu,\lambda}$ and if e meets $S_{\beta,\alpha}$, $(\alpha, \beta) = \mathcal{O}$, then $S_{\mu,\lambda} = S_{\beta,\alpha}$ so $\alpha/\beta = \lambda/\mu$.*

This is clear because $S_{\beta,\alpha}$ is a union of open geodesics passing through some chosen $p \in e \cap S_{\beta,\alpha}$ (i.e., take $S_{\beta,\alpha} \cap P$ for all vertical planes P

through p). None of these can pierce $S_{\mu,\lambda}$ otherwise they would meet int B.

COROLLARY 4.6. *If $p \in B$, then*

(1) $d(p) = 0$ *if and only if* $p \in \operatorname{int} B$
(2) $d(p) = 1$ *if and only if p lies in some open 2-cell of ∂B.*

This follows from the preceding Corollary and Lemma 4.2.

COROLLARY 4.7. *Let e be an open edge of ∂B. Let $(\mu, \lambda) = \mathcal{O}$. Let $p \in e$. If $p \in S_{\mu,\lambda}$ then $e \subset S_{\mu,\lambda}$.*

We need only check that e cannot pierce $S_{\mu,\lambda}$. If it did, some point of $S_{\mu,\lambda}$ would be directly over some point of e. Thus $S_{\mu,\lambda}$ would meet int B which is impossible.

COROLLARY 4.8. *If e is an open edge of ∂B, then $d(p)$ is constant for $p \in e$.*

This follows from the previous corollary and Lemma 4.2.

LEMMA 4.9. *Let $\sigma \in SL(2, \mathcal{O})$. Let e be an open n-cell of ∂B, $n = 0, 1, 2$. If σe meets B then σe is also an open n-cell of ∂B.*

Proof. Suppose first that $n = 2$. Let $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \mu & \lambda \end{smallmatrix}\right)$. Since $e \subset B$ and σe meets B, we see that e meets $B \cap \sigma^{-1}B = B \cap S_{\mu,\lambda}$. By Corollary 4.5, e must be the 2-cell of B with $\bar{e} = B \cap S_{\mu,\lambda} = B \cap \sigma^{-1}B$. Now $\sigma\bar{e} = \sigma B \cap B = B \cap S_{\mu,-\alpha}$ since $\sigma^{-1} = \left(\begin{smallmatrix} \lambda & -\beta \\ -\mu & \alpha \end{smallmatrix}\right)$. Let e' be the open cell of ∂B with $\bar{e}' = B \cap S_{\mu,-\alpha}$. Then $\sigma\bar{e} = \bar{e}'$ so $\sigma e \subset e'$. Applying the same argument to σ^{-1} and e' we see that $\sigma^{-1}e' \subset e$ so $\sigma e = e'$.

Now suppose e is an edge. Again, e meets $B \cap \sigma^{-1}B = B \cap S_{\mu,\lambda}$ so $e \subset S_{\mu,\lambda}$ by Corollary 4.7. Therefore $e \subset B \cap \sigma^{-1}B$ so $\sigma e \subset B$. Since $d(p) > 1$ for all $p \in e$, and $d(\sigma p) = d(p)$, we see that σe lies in the 1-skeleton of ∂B (using Corollary 4.6). I claim that no vertex of ∂B lies on σe. This is clear if v is a singular point. Suppose v is a nonsingular vertex lying on σe. The part of σe lying near v and on one side of v must lie on some open edge e' with one end at v. Since v is a vertex there is also some other edge e'' with one end at v such that e'' is not simply a continuation of e' (i.e., $e' \cup e''$ does not lie on a geodesic). Now by Corollary 4.8, d is constant on e' and also on e. Since d is invariant under σ, it is constant on σe. Since $v \in \sigma e$ and σe meets e' we see that $d(v) = d(p)$

for all $p \in e'$. By Corollary 4.7, $d(p)$ is the number of $S_{\mu,\lambda}$ such that $e' \subset S_{\mu,\lambda}$. Since $v \in \bar{e}'$ and $e' \subset S_{\mu,\lambda}$ implies $\bar{e}' \subset S_{\mu,\lambda}$, we see that every $S_{\mu,\lambda}$ containing v also contains e'. But e'' lies on some intersection $S_{\mu,\lambda} \cap S_{\beta,\alpha}$ and $v \in \bar{e}''$. Therefore e' and e'' both lie on $S_{\mu,\lambda} \cap S_{\beta,\alpha}$, so $e' \cup e''$ is a geodesic arc, contradicting our choice of e''.

This shows that σe is a connected subset of the 1-skeleton of ∂B which contains no vertex. Therefore, there is some edge e' of ∂B with $\sigma e \subset e'$. Applying the same argument to $\sigma^{-1}e'$, we see that $\sigma^{-1}e' \subset e''$ for some open edge e''. Since $e \subset e''$, $e = e''$ and $\sigma e = e'$.

Finally, if e is a vertex, σe cannot lie in an open 1 or 2-cell by the above results applied to σ^{-1}. Therefore σe must be a vertex.

Definition. Let E be the set of $p \in H$ with $d(p) > 1$ and let $E_1 = B \cap E$.

By Corollary 4.6, E_1 is the 1-skeleton of ∂B. Clearly E is the union of all σE_1 for $\sigma \in SL(2, \mathcal{O})$. Since E_1 is stable under Φ, we see that $E_1 = \bigcup \sigma E_1'$ over $\sigma \in \Phi$ where E_1' is the finite subcomplex of E_1 consisting of those closed edges which meet the domain D. Therefore $E = \bigcup \sigma E_1'$ over all $\sigma \in SL(2, \mathcal{O})$. By Proposition 3.9, E is a locally finite union of closed geodesic segments of the form $\sigma\bar{e}$ where $\bar{e}$ runs over the closed edges of B.

Lemma 4.10. *If e, e' are open edges of B and $\sigma, \tau \in SL(2, \mathcal{O})$ then either $\sigma\bar{e} = \tau\bar{e}'$, $\sigma\bar{e} \cap \tau\bar{e}' = \emptyset$, or $\sigma\bar{e} \cap \sigma\bar{e}' = \{p\}$ where p is an endpoint of $\sigma\bar{e}$ and of $\tau\bar{e}'$.*

Proof. Suppose $\sigma\bar{e} \neq \tau\bar{e}'$ and $p \in \sigma\bar{e} \cap \tau\bar{e}'$. Suppose $p \in \sigma e$. Then $\tau^{-1}\sigma e$ meets $\bar{e}' = \tau^{-1}\tau\bar{e}'$ at $\tau^{-1}p$. By Lemma 4.9, $\tau^{-1}\sigma e = e''$ is an open edge of ∂B and so cannot meet $\bar{e}'$. Therefore we cannot have $p \in \sigma e$ or $p \in \sigma e'$. Finally if $\sigma\bar{e}$ and $\tau\bar{e}'$ meet at both ends then $\sigma\bar{e} = \tau\bar{e}'$ by the uniqueness of the geodesic through two given points.

Corollary 4.11. *The set E has a regular cell subdivision. Its edges are the σe where e runs over the edges of ∂B and $\sigma \in SL(2, \mathcal{O})$. Its vertices are the σv where v runs over the vertices of ∂B.*

Now, let $X = H - E$ and $A = B - E = B - E_0$. We will apply Corollary 1.7 to this situation to obtain a presentation for $G = SL(2, \mathcal{O})$. Clearly X is connected and locally pathwise connected. The argument showing that B is pathwise connected also applies to A since $E_0 \subset \partial B$. Let $D' = D - E$. Then $B = \Phi D'$. Since D is G-normal in H, D' is G-normal in X. Since H has a G-invariant metric, so does X. By

Theorem 2 of [13], D' is G-inflatable and so, by Lemma 1.8, A is also G-inflatable. Consequently, we can apply Corollary 1.7. Furthermore, Lemma 1.9 also applies. For, suppose $A \cap \sigma A \cap \tau A \neq \emptyset$. Let

$$\sigma^{-1} = \begin{pmatrix} \lambda_1 & \mu_1 \\ \mu & \lambda \end{pmatrix} \quad \text{and} \quad \tau^{-1} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \beta & \alpha \end{pmatrix}.$$

If $\mu = 0$, $\sigma A = A$. Suppose $\mu \neq 0$. Then $B \cap \sigma B = B \cap S_{\mu,\lambda}$ and $B \cap \tau B = B \cap S_{\beta,\alpha}$ so $B \cap \sigma B \cap \tau B = B \cap S_{\mu,\lambda} \cap S_{\beta,\alpha}$. If $S_{\mu,\lambda} \neq S_{\beta,\alpha}$, this set lies in E by Corollaries 4.5 and 4.6. Therefore $A \cap \sigma A \cap \tau A = \emptyset$ unless $S_{\mu,\lambda} = S_{\beta,\alpha}$. In this case $\Phi\sigma^{-1} = \Phi\tau^{-1}$ by Lemma 4.3 so $\sigma B = \tau B$ and $\sigma A = \tau A$.

To apply Lemma 1.9, we must first determine the set S of all $\sigma \in SL(2, \mathcal{O})$ with $\sigma A \cap A \neq \emptyset$. If $\sigma = \left(\begin{smallmatrix} \lambda_1 & \mu_1 \\ \mu & \lambda \end{smallmatrix}\right)$, we have $\sigma A \cap A \neq \emptyset$ if and only if $\sigma B \cap B \not\subset E$ or $B \cap \sigma^{-1} B = B \cap S_{\mu,\lambda} \not\subset E$. The last condition is equivalent to the assertion that $B \cap S_{\mu,\lambda}$ is a 2-cell of ∂B. Therefore S is the set of $\left(\begin{smallmatrix} \lambda_1 & \mu_1 \\ \mu & \lambda \end{smallmatrix}\right) \in SL(2, \mathcal{O})$ such that either $\mu = 0$ or $B \cap S_{\mu,\lambda}$ is a 2-cell of ∂B.

Next we choose a set of representatives $\sigma_0 = 1, \sigma_1, \ldots, \sigma_n$ for the double cosets $\Phi\sigma\Phi$ which make up S. By Lemma 4.3 this can be done as follows. First choose a set of representatives $\alpha_1, \ldots, \alpha_n \in F$ for the λ/μ with $B \cap S_{\mu,\lambda}$ a 2-cell of ∂B, modulo the equivalence relation $\alpha \sim \beta$ if $\beta = u^2\alpha + \gamma$ for $u \in \mathcal{O}^*$, $\gamma \in \mathcal{O}$. Note that except for $K = Q(\sqrt{-1})$ and $Q(\sqrt{-3})$, we have $u^2 = 1$ so $\alpha \sim \beta$ if and only if $\alpha \equiv \beta \bmod \mathcal{O}$. An equivalent way to choose the α_i is to choose a set of representatives $B \cap \partial S_{\mu_i}$, for the 2-cells of ∂B modulo the action of the group Φ. We now set $\alpha_i = \lambda_i/\mu_i$, $(\mu_i, \lambda_i) = \mathcal{O}$ and choose some $\sigma_i = A_i = \left(\begin{smallmatrix} \lambda_i' & \mu_i' \\ \mu_i & \lambda_i \end{smallmatrix}\right)$. To avoid unnecessary subscripts in the presentations obtained later, I will denote the generators of $SL(2, \mathcal{O})$ by capital roman letters rather than by σ_i, etc. This can hardly cause any confusion with the sets A, B, etc.

We now choose a set of generators for the group Φ. Let $T_s = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)$ for $s \in \mathcal{O}$. Let $1, \omega$ be a base for $\mathcal{O}$ as Abelian group and let $T = T_1$, $U = T_\omega$. For $\rho \in \mathcal{O}^*$, let $L_\rho = \left(\begin{smallmatrix} \rho^{-1} & 0 \\ 0 & \rho \end{smallmatrix}\right)$. Clearly Φ is a split extension with kernel the subgroup consisting of all T_s and complementary subgroup the subgroup consisting of the L_ρ. In particular, every element of Φ has uniquely the form $L_\rho T_s$, $\rho \in \mathcal{O}^*$, $s \in \mathcal{O}$.

Definition. If $\sigma \in S$ and $\sigma \notin \Phi$ we can write σ in the form $\sigma = L_\rho T_s A_i L_\nu T_t$ with $\rho, \nu \in \mathcal{O}^*$, $s, t \in \mathcal{O}$. We refer to this as the canonical form for σ and denote the formal word $L_\rho T_s A_i L_\nu T_t$ by $[\sigma]$. Whenever possible we choose $\nu = 1$.

In general, there will be some ambiguity in ρ, s, ν, t but we assume some definite choice has been made. Of course, we should write $[\sigma]$ as $[L_\rho][T_s][A_i][L_\nu][T_t]$ using the notation of Section 1 but I will drop the extra brackets, it being understood that $[\sigma]$ represents a formal word in the generators. We should also write out each T_s in terms of T and U since these are the only T_s which will appear in our list of generators.

LEMMA 4.12. *The group Γ of Lemma 8 for the situation considered above is generated by the elements T, U, L_ρ with $\rho \in \mathcal{O}^*$, and $A_1, \ldots, A_n$ with the relations*

(1) $TU = UT$,
(2) $L_\rho L_\nu = L_{\rho\nu}$ *for all* $\rho, \nu \in \mathcal{O}^*$,
(3) $L_\rho^{-1} T_s L_\rho = T_{s\rho^2}$ *for* $s = 1, \omega$,
(4) $A_i^{-1} = [A_i^{-1}]$ *for* $i = 1, \ldots, n$,
(5) $A_i L_\rho = [A_i L_\rho]$ *for all* $i = 1, \ldots, n$, $\rho \in \mathcal{O}^*$ *which satisfy* $\rho^2 \equiv 1 \bmod \mu_i$.

In the relations (5) it is important to note that in our definition of canonical form we have chosen to have $\nu = 1$ whenever possible.

Proof. The relations (1), (2), (3) give a presentation for Φ as a split extension as noted above. It is trivial to verify (3) for all $s \in \mathcal{O}$ but the cases $s = 1, \omega$ suffice to determine the action of the L_ρ on the T_s. Therefore the relations (1), (2), (3) correspond to the relations (a) of Lemma 1.9. The relations (4) are the relations (b) of Lemma 1.9 in condensed form. It remains to check that the relations (5) are equivalent to the relations (c) of Lemma 1.9. Let $A = \left(\begin{smallmatrix} \alpha & \beta \\ \mu & \lambda \end{smallmatrix}\right)$. An easy calculation shows that

$$L_\nu T_s A L_\rho T_t = \begin{pmatrix} \rho^{-1}\nu^{-1}\alpha + \rho^{-1}\nu^{-1}s\mu & * \\ \rho^{-1}\nu\mu & \rho\nu\lambda + \rho^{-1}\nu t\mu \end{pmatrix}.$$

We can ignore the entry * since a matrix in $SL(2, \mathcal{O})$ is determined by three of its entries. For this matrix to be A we must have $\rho = \nu$, $\lambda = \rho^2\lambda + t\mu$, and $\alpha = \rho^{-2}\alpha + \rho^{-2}s\mu$ or $\rho^2\alpha = \alpha + s\mu$. Since $(\mu, \lambda) = \mathcal{O}$, $\lambda = \rho^2\lambda + t\mu$ implies $\rho^2 \equiv 1 \bmod \mu$. Conversely, if $\rho^2 \equiv 1 \bmod \mu$, we can solve for s and t. The relations (c) of Lemma 1.9 arise from the nontrivial relations of the form $A = L_\rho T_s L_\nu T_t$ where A is one of the A_i. We have just seen that such a relation holds if and only if $\rho^2 \equiv 1 \bmod \mu$.

If so the relation is equivalent to $AL_\rho = T_{-s}L_\nu^{-1}AT_{-t} = L_{\nu^{-1}}T_{-s\rho^{-2}}AT_{-t}$. Since our argument above shows that ν, s, t are uniquely determined by A and ρ, the right side of this relation is the unique canonical form for AL_ρ.

If we exclude the cases $K = Q(\sqrt{-1})$ and $K = Q(\sqrt{-3})$, the only units of $\mathcal{O}$ are ± 1 and the only L_ρ we need consider is $J = \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$.

COROLLARY 4.13. *If* $K \neq Q(\sqrt{-1})$ *or* $Q(\sqrt{-3})$, Γ *is generated by the elements* J, T, U, $A_1, \ldots, A_n$ *with the relations*

(1) $TU = UT$,
(2) $J^2 = 1$,
(3) J *is central*,
(4) $A_i^{-1} = [A_i^{-1}]$.

In fact the relations (3) and (5) of Lemma 4.12 are covered by the assertion that J is central. It is convenient to keep J as a generator even if it can be expressed in terms of the other generators because the center of $SL(2, \mathcal{O})$ is $\{1, J\}$.

It is also worth noting that for $K \neq Q(\sqrt{-1})$ or $Q(\sqrt{-3})$ the canonical form is unique because we can always take $\nu = 1$ and

$$L_\rho T_s A T_t = \begin{pmatrix}\rho^{-1}\alpha + \rho^{-1}s\mu & * \\ \rho\mu & \rho\lambda + \rho t\mu\end{pmatrix}.$$

This determines ρ, s, and t uniquely for $\mu \neq 0$.

It is convenient to choose the A_i so that $A_i^{-1} = A_j$ for some j as often as possible. If this is done all relations (4) take the form $A_i^{-1} = A_j$ except when $A_i^{-1} = L_\rho T_s A_i L_\nu T_t$.

To get a presentation for $G = SL(2, \mathcal{O})$ we must now calculate the relations $\theta(\lambda_i) = 1$ of Corollary 1.7. Since H is topologically Euclidean 3-space and E is a 1-complex, we can apply Corollary 1.11 to compute $\pi_1(X) = \pi_1(H - E)$. From this we deduce that $\pi_1(X)$ is generated normally by small circular loops λ_e, one around each edge e of E. There are a finite number of edges $e_1, \ldots, e_r$ (e.g., a subset of those edges meeting D) such that every edge e has the form $e = \sigma e_i$ for $\sigma \in G$, $i = 1, \ldots, r$. We let $\lambda_i = \lambda_{e_i}$ and choose $\lambda_e = \sigma\lambda_i$. Clearly, $\pi_1(X)$ is normally generated over G by the loops $\lambda_1, \ldots, \lambda_r$, so G is obtained from Γ by adding the relations $\theta(\lambda_i) = 1$, $i = 1, \ldots, r$.

We must now calculate $\theta(\lambda)$ where λ is a small loop about the edge e. Let (z_0, ζ_0) be a point of the open edge e. Let $u \in \mathbf{C}$ with $|u| = 1$

such that the line from 0 to u is parallel to the projection of e on $\mathbf{C}$ (identified with the plane $\zeta = 0$). For λ we choose the circle given by $z = z_0 + i\epsilon u \cos\theta$, $\zeta = \zeta_0 + \epsilon \sin\theta$, where $0 \leqslant \theta \leqslant 2\pi$ and $\epsilon > 0$ is a fixed small real number. We can take ϵ as small as is necessary because the loops λ so defined are all homotopic if ϵ is so small that λ links no other edge of E. It is clear that λ is well-behaved because the sets σB are all bounded by planes and spheres. This will also appear directly from our calculations.

To begin with, by Proposition 3.9 we can take ϵ so small that λ does not meet any τD which does not contain the point $p = (z_0, \zeta_0)$. Thus we may restrict our attention to the τ with $p \in \tau^{-1}B$. By Lemma 4.9 this condition is equivalent to $e \subset \tau^{-1}B$ and implies that τe is an edge of B. If $p \in \tau^{-1}B$ then $\tau p \in B$ and there is some $\varphi \in \Phi$ with $\varphi\tau p \in D$. By Proposition 3.9, there are only a finite number of possibilities for $\varphi\tau$ so the $\tau \in G$ with $\tau p \in B$ form a finite union of cosets $\bigcup \Phi\tau_i$ for $i = 0,\ldots, s$, say, with $\tau_0 = 1$. Since $\tau^{-1}B = \tau_i^{-1}B$ for $\tau \in \Phi\tau_i$ it will suffice to find, for each $(z, \zeta) \in \lambda$, those $i = 0,\ldots, s$ with $\tau_i(z, \zeta) \in B$. Note that $\lambda \subset \bigcup \tau_i B$ by the choice of ϵ and the τ_i.

Lemma 4.14. *For each $i = 0,\ldots, s$, $\lambda \cap \tau_i^{-1} \operatorname{int} B \neq \emptyset$ provided ϵ is small enough.*

Proof. For each such i, $\tau_i e$ is an open edge e_i of ∂B. Let S_i consist of points (z, ζ) such that there is some $(z, \zeta') \in e_i$ with $\zeta' \leqslant \zeta$. Topologically, S_i is a rectangle, open on 3 sides, with one side e_i, and $S_i - e_i \subset \operatorname{int} B$. Now $\tau_i^{-1}S_i$ is topologically an open rectangle with one side e and $\tau_i^{-1}S_i - e \subset \operatorname{int} \tau_i B$. If ϵ is small enough, λ must pierce this rectangle and so must meet $\operatorname{int} \tau_i B$.

Lemma 4.15. *If $i \neq j$, $\lambda \cap \tau_i^{-1}B \cap \tau_j^{-1}B$ is a finite set, and so is each $\lambda \cap \tau_i^{-1}\partial B$.*

Proof. If $\sigma B \neq B$, $\sigma \notin \Phi$ so $\sigma = \left(\begin{smallmatrix}\lambda_1 & \mu_1\\ \mu & \lambda\end{smallmatrix}\right)$ with $\mu \neq 0$. Therefore $B \cap \sigma^{-1}B = B \cap S_{\mu,\lambda} \subset \partial B$. Therefore $\tau_i^{-1}B \cap \tau_j^{-1}B \subset \tau_i^{-1}\partial B$. This is contained in a union of hemispheres $\tau_i^{-1}S_{\beta,\alpha}$ and λ can meet only a finite number of these by Theorem 3.13. If $\lambda \cap \tau_i^{-1}\partial B$ is infinite, λ meets one of these hemispheres, say, $\tau_i^{-1}S_{\beta,\alpha}$, in an infinite set and so $\lambda \subset \tau_i^{-1}S_{\beta,\alpha}$ since λ is a circle. Therefore λ cannot meet $\tau_i^{-1} \operatorname{int} B$ contradicting Lemma 4.14.

Lemma 4.16. *For each $i = 1,\ldots, s$, $\lambda \cap \tau_i^{-1}B$ is either empty or an arc of λ.*

Proof. We call a subset C of H convex if for each $p, q \in C$, the closed geodesic segment from p to q lies in C. Clearly, the intersection of convex sets is convex. If S is a hemisphere in H with center in the plane $\zeta = 0$, Lemma 4.4 shows that the set of points of H lying outside or on S is convex. Therefore B is convex, being an intersection of such sets. Since each $\tau \in G$ is an isometry, τB is convex.

Let C be the disk consisting of all (z, ζ) with $z = z_0 + iru \cos \theta$, $\zeta = \zeta_0 + r \sin \theta$ for $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant r \leqslant \epsilon$. Thus C is the Euclidean disk with boundary λ and lies in the vertical plane P defined by $z = z_0 + iut$, t running over all real numbers. If l is any geodesic not lying in P, then $P \cap l$ has at most one point. If l lies in P, then clearly $l \cap C$ is either empty, one point, or an arc of l since we are just looking at the intersection of two circles in the plane. It follows that C is also convex.

Suppose X is a convex set, $p \in X$, and $q \in \operatorname{int} X$. Then the open geodesic segment (p, q) lies in $\operatorname{int} X$ because if $r \in (p, q)$ and $s \in H$ is sufficiently close to r, the geodesic through p and r lies so close to the geodesic through p and q that it meets $\operatorname{int} X$ near q. By the convexity of X, $s \in X$ so X contains a neighborhood of r.

Suppose now that $p \in \lambda \cap \tau_i^{-1}B$ and $q \in \tau_i^{-1} \operatorname{int} B$. There is such a q by Lemma 4.14. Let l be the half-open geodesic segment $(p, q]$. Then $(p, q] \subset \tau_i^{-1} \operatorname{int} B$ by the preceding remark. Suppose neither of the two arcs from p to q on λ lies in $\tau_i^{-1}B$. Choose a point on each of these two arcs not in $\tau_i B$. This gives us two points $r, s \in \lambda$ which separate p from q on λ with $r, s \notin \tau_i B$. Now $C \subset \bigcup \tau_j^{-1}B$, so for some $j, k \neq i$ we have $r \in C \cap \tau_j^{-1}B$, $s \in C \cap \tau_k^{-1}B$. These are convex. Also the center $p_0 = (z_0, \zeta_0)$ of C lies on e and so is in all $\tau_j^{-1}B \cap C$. Join r to p_0 in $C \cap \tau_j^{-1}B$ and p_0 to s in $C \cap \tau_k^{-1}B$ by geodesics. This gives a curve ω joining r to s in C, disjoint from $\tau_i^{-1} \operatorname{int} B$, and meeting λ only at r and s. Since r, s separate p, q on λ, ω must meet (p, q) which is impossible.

From these results we see that the loop λ is subdivided into non-overlapping nontrivial arcs $\lambda \cap \tau_i^{-1}B$. In particular, we see that λ is well-behaved. To compute $\theta(\lambda)$ we must determine the order in which the $\tau_i^{-1}B \cap \lambda$ occur on λ. The edge e lies on a geodesic g which is a semicircle. Let $(z_1, 0)$ be its center. Let l be the line in $\mathbf{C}$ through z_0 perpendicular to the projection of e on $\mathbf{C}$. If e lies on $S_{\mu,\lambda}$, $\mu \neq 0$, the center λ/μ of $S_{\mu,\lambda}$ must lie on the line l. If $\tau = \left(\begin{smallmatrix} \lambda_1 & \mu_1 \\ \mu & \lambda \end{smallmatrix}\right)$, $\mu \neq 0$ then $e \subset \tau^{-1}B$ if and only if $e \subset B \cap \tau^{-1}B = B \cap S_{\mu,\lambda}$, i.e., if and only if e lies on $S_{\mu,\lambda}$. By Lemma 4.3 the coset $\Phi\tau$ is uniquely determined by the point $\alpha_i = \lambda_i/\mu_i$. Thus the τ_i considered above except for $\tau_0 = 1$ correspond uniquely to the $\alpha_i \in l$ which are centers of $S_{\mu,\lambda}$ containing e (with $(\mu, \lambda) = \mathcal{O}$). We think of $\tau_0 = 1$ as corresponding to a point at infinity on l.

LEMMA 4.17. *The arcs $\lambda \cap \tau_i^{-1}B$ occur on λ in the same order in which the points α_i (with $\alpha_0 = \infty$) occur on l (for small ϵ).*

Proof. Let $(z, \zeta) \in \lambda$. We want to find the ν with $(z, \zeta) \in \tau_\nu^{-1}B$, i.e., $\tau_\nu(z, \zeta) \in B$. Let $\tau_\nu = \begin{pmatrix} \lambda_\nu' & \mu_\nu' \\ \mu_\nu & \lambda_\nu \end{pmatrix}$ and $\tau_\nu(z, \zeta) = (z_\nu, \zeta_\nu)$. By Lemma 3.1,

$$\zeta_i = \frac{\zeta}{|\mu_\nu z - \lambda_\nu|^2 + \zeta^2 |\mu_\nu|^2}.$$

By Lemma 3.6, $(z_\nu, \zeta_\nu) \in B$ if and only if ζ_ν is maximal among the ζ_j. Thus we must find those i which minimize $|\mu_\nu z - \lambda_\nu|^2 + \zeta^2 |\mu_\nu|^2$. We temporarily drop the subscript ν for convenience. Since $(z_0, \zeta_0) \in e \subset \tau^{-1}B$, we have $|\mu z_0 - \lambda|^2 + \zeta_0^2 |\mu|^2 = 1$. Now $z = z_0 + iu\epsilon \cos\theta$, $\zeta = \zeta_0 + \epsilon \sin\theta$, so

$$\begin{aligned} |\mu z - \lambda|^2 + \zeta^2 |\mu|^2 &= |\mu z_0 - \lambda|^2 + \zeta_0^2 |\mu|^2 + 2R((\bar\mu \bar z_0 - \bar\lambda)\, \mu i u \epsilon \cos\theta) \\ &\qquad + 2\zeta_0 \epsilon \sin\theta\, |\mu|^2 + \epsilon^2 \cos^2\theta\, |\mu|^2 + \epsilon^2 \sin^2\theta\, |\mu|^2 \\ &= 1 + 2R((\bar\mu \bar z_0 - \bar\lambda)\, \mu i u \epsilon \cos\theta) \\ &\qquad + 2\zeta_0 \epsilon \sin\theta\, |\mu|^2 + \epsilon^2 |\mu|^2. \end{aligned}$$

Now the edge e can be parametrized by $z = z_1 + ur\cos\varphi$, $\zeta = r\sin\varphi$ where r is the radius of the circle on which e lies, $(z_1, 0)$ is its center, and φ ranges over some open interval. Since $(z_0, \zeta_0) \in e$, there is a φ_0 with $z_0 = z_1 + ur\cos\varphi_0$, $\zeta_0 = r\sin\varphi_0$. Now e lies on $S_{\mu,\lambda}$, so $|\mu z - \lambda|^2 + \zeta^2 |\mu|^2 = 1$ for $(z, \lambda) \in e$, i.e.,

$$\begin{aligned} |\mu z_1 + \mu u r \cos\varphi - \lambda|^2 + |\mu|^2 r^2 \sin^2\varphi = 1 &= |\mu z_1 - \lambda|^2 \\ + 2R((\bar\mu \bar z_1 - \bar\lambda)\, \mu u r \cos\varphi) &+ |\mu|^2 r^2 \cos^2\varphi + |\mu|^2 r^2 \sin^2\varphi. \end{aligned}$$

Thus $R((\bar\mu \bar z_1 - \bar\lambda)\, \mu u r \cos\varphi)$ is independent of φ. Therefore it is zero, so $(\bar\mu \bar z_1 - \bar\lambda)\mu u$ is purely imaginary. Now

$$(\bar\mu \bar z_0 - \bar\lambda)\, \mu i u = \bar\mu(\bar z_0 - \bar z_1)\, \mu i u + (\bar\mu \bar z_1 - \bar\lambda)\, \mu i u.$$

The second term is real. The first is $|\mu|^2 iu(\bar u r \cos\varphi_0)$ which is purely imaginary.

Returning now to the point $(z, \zeta) \in \lambda$, we have

$$\begin{aligned} |\mu z - \lambda|^2 + \zeta^2 |\mu|^2 &= 1 + 2R((\bar{\mu}\bar{z}_0 - \bar{\lambda})\,\mu iu\epsilon \cos\theta) + 2\zeta_0 |\mu|^2 + \epsilon^2 |\mu|^2 \\ &= 1 + 2(\bar{\mu}\bar{z}_1 - \bar{\lambda})\,\mu iu\epsilon \cos\theta + 2\zeta_0\epsilon |\mu|^2 + \epsilon^2 |\mu|^2. \end{aligned}$$

Also $(\bar{\mu}\bar{z}_1 - \lambda)\,\mu iu = -i\bar{u}\bar{\mu}(\mu z_1 - \lambda)$ since it is real. Let $\alpha = \lambda/\mu$. Then $|\mu z - \lambda|^2 + \zeta^2 |\mu|^2 = 1 + |\mu|^2[2\epsilon(\zeta_0 \sin\theta - i\bar{u}(z_1 - \alpha)\cos\theta) + \epsilon^2]$. If $\mu = 0$, we get instead 1. We must now find the ν such that $\mu = \mu_\nu$, $\lambda = \lambda_\nu$ minimizes the expression. To find the order in which the intervals occur, it will suffice to do this for one (z, ζ) in each of the open arcs $\zeta \cap \tau^{-1}\operatorname{int} B$. For each of these there is a unique ν which gives the minimum. If ϵ is sufficiently small, the term in ϵ^2 will not change the value of ν for which this minimum is attained. Thus we can ignore the ϵ^2. Of course we could also obtain an alternative proof of Lemma 4.17 by analyzing the exact expression with the ϵ left in.

Our problem is now to find the ν which minimizes

$$|\mu_\nu|^2 (\zeta_0 \sin\theta - i\bar{u}(z_1 - \alpha_\nu)\cos\theta).$$

For $\nu = 0$ this is replaced by 0. Write

$$|\mu_\nu|^2 (\zeta_0 \sin\theta - i\bar{u}(z_1 - \alpha_\nu)\cos\theta) = c_\nu \cos\theta + d_\nu \sin\theta$$

for convenience, where $c_0 = d_0 = 0$. Let p_ν be the point in the plane with coordinates $(c_\nu, -d_\nu)$. If R_θ denotes the rotation about $(0, 0)$ through an angle θ, $R_\theta p_\nu$ has x-coordinate $c_\nu \cos\theta + d_\nu \sin\theta$. Therefore we are looking for the ν for which $R_\theta p_\nu$ lies furthest to the left in the plane. Let C be the convex hull of the points p_j. Then $R_\theta p_\nu$ is still the farthest point to the left in the set $R_\theta C$. Therefore as θ runs from 0 to 2π, the points p_ν with $R_\theta p_\nu$ furthest to the left are, in turn, the points p_ν which lie on the boundary ∂C. Since every p_ν must have its turn by Lemma 4.14, all p_ν lie on ∂C. Since C is convex, we see that if we start with p_0, the order of the remaining points on ∂C is determined by the angle φ_ν between the line $p_0 p_\nu$ and the x axis. Since all $c_\nu > 0$ for $\nu \neq 0$, $-\pi/2 < \varphi_\nu < \pi/2$ and the p_ν occur in the same order as the values of $\tan\varphi_\nu = -d_\nu/c_\nu = \zeta_0^{-1} iu(z_1 - \alpha_\nu)$. But this is just the order of the points α_ν on the line l. This proves the Lemma.

There are of course two possible orders depending on how we orient l and λ. There is no need to specify which one we choose since a change of orientation only changes $\theta(\lambda)$ to $\theta(\lambda^{-1}) = \theta(\lambda)^{-1}$ (up to conjugation).

Now if we choose $\tau_1 = 1$, $\tau_2, \ldots, \tau_s$ so that the corresponding $\alpha_1 = \infty$, $\alpha_2, \ldots, \alpha_s$ lie in order on l, we get

$$\theta(\lambda) = [\tau_0\tau_1^{-1}][\tau_1\tau_2^{-1}] \cdots [\tau_{s-1}\tau_s^{-1}][\tau_s\tau_0^{-1}] = [\tau_1^{-1}][\tau_1\tau_2^{-1}] \cdots [\tau_{s-1}\tau_s^{-1}][\tau_s]$$

because the arc $\lambda \cap \tau_i^{-1}B$ lies in $\tau_i^{-1}B$ so we have $\rho_i = \tau_i^{-1}$ in the definition of θ. Since all $\tau_i^{-1}\tau_{i+1} \in S$, $\tau_i^{-1}\tau_{i+1}$ has a canonical form $[\tau_i^{-1}\tau_{i+1}]$ expressing it in terms of our chosen generators.

Summary

We now summarize our results for the case $K \neq Q(\sqrt{-1})$ or $Q(\sqrt{-3})$. These two exceptional cases will be treated in the next two sections. Let $K = Q(\sqrt{-m})$ be an imaginary quadratic number field not $Q(\sqrt{-1})$ or $Q(\sqrt{-3})$ with ring of integers $\mathcal{O}$. We assume for the moment that we know the cell structure of ∂B. The problem of determining this will be discussed later.

The cell structure of ∂B is stable under translations by elements of $\mathcal{O}$. Choose a set of representatives e_i, $i = 1, \ldots, n$ for the 2-cells of ∂B modulo translations by $\mathcal{O}$. Let $\bar{e}_i = B \cap S_{\mu_i,\lambda_i}$ and choose some

$$A_i = \begin{pmatrix} \lambda_i' & \mu_i' \\ \mu_i & \lambda_i \end{pmatrix} \in SL(2, \mathcal{O}).$$

Let $1, \omega$ be a base for $\mathcal{O}$ as Abelian group, let $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $U = \left(\begin{smallmatrix} 1 & \omega \\ 0 & 1 \end{smallmatrix}\right)$. Let $T_{m+n\omega} = T^m U^n$. Let $J = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. If an element $\sigma \in SL(2, \mathcal{O})$ can be written in the form $\sigma = J^\epsilon T_s A_i T_t$ with $\epsilon = 0, 1$, $s, t \in \mathcal{O}$, $i = 1, \ldots, n$, then this representation is unique. We call it the canonical form for σ and write $[\sigma]$ for the formal word $J^\epsilon T_s A_i T_t$.

If e is an edge of ∂B, let S_{μ_i,λ_i} be the distinct $S_{\mu,\lambda}$ with $(\mu, \lambda) = \mathcal{O}$ such that e lies on $S_{\mu,\lambda}$. The $\alpha_i = \lambda_i/\mu_i$ all lie on a line. We number the α_i so they lie on the line in the order $\alpha_1, \alpha_2, \ldots, \alpha_s$. Choose

$$\tau_i = \begin{pmatrix} \lambda_i' & \mu_i' \\ \mu_i & \lambda_i \end{pmatrix} \in SL(2, \mathcal{O}).$$

Let R_e be the formal word $[\tau_1^{-1}][\tau_1\tau_2^{-1}] \cdots [\tau_{s-1}\tau_s^{-1}][\tau_s]$. This depends on the orientation of the line and the choice of the τ_i but this ambiguity will not affect the final result. We assume some definite choice has been made. The relation $R_e = 1$ obviously holds in $SL(2, \mathcal{O})$. We refer to it as the edge relation corresponding to e. There is an automatic check here because if we omit some τ_i or take them in the wrong order, some $\tau_{i-1}\tau_i^{-1}$ will not have a canonical form.

THEOREM 4.18. *Let K be a quadratic imaginary number field not $Q(\sqrt{-1})$ or $Q(\sqrt{-3})$ with ring of integers $\mathcal{O}$. Then $SL(2, \mathcal{O})$ is generated by T, U, J, $A_1, \ldots, A_n$ with the relations*

(1) $TU = UT$, $J^2 = 1$, *J central,*

(2) $A_i^{-1} = [A_i]$, $i = 1, \ldots, n$,

(3) *The edge relations $R_{e_i} = 1$ where e_i runs over a set of representatives of the edges of ∂B modulo the translations by elements of $\mathcal{O}$.*

This follows immediately from Corollary 1.7, Corollary 4.13, and the results obtained above.

To conclude this section, we give some comments on the calculation of the edge relations. If e is an edge of ∂B, e lies on a semicircle C with center $(z_0, 0)$, say. The centers α_i of the S_{μ_i,λ_i} containing e all lie on the line l through z_0 and perpendicular to the projection of e on $\mathbf{C}$. Let (z_0, ζ_0) be the point on C lying over z_0, i.e., the highest point on C. We refer to (z_0, ζ_0) as the apex of e. Note that conceivably (z_0, ζ_0) may not lie on e itself.

LEMMA 4.19. *If $(\mu, \lambda) = \mathcal{O}$, e lies on $S_{\mu,\lambda}$ if and only if the apex of e lies on $S_{\mu,\lambda}$ and the center of $S_{\mu,\lambda}$ lies on l.*

Proof. If e lies on $S_{\mu,\lambda}$, the whole semicircle C lies on $S_{\mu,\lambda}$ and conversely. If the center of $S_{\mu,\lambda}$ lies on l, $S_{\mu,\lambda}$ meets the plane of C in a semicircle whose apex lies over z_0. Since this semicircle and C have the same center, they meet if and only if they have the same apex, when they coincide.

Remark. Instead of the apex we can clearly use any point p of C having $\zeta > 0$. If p lies on the open edge e we can even omit the requirement that $\lambda/\mu \in l$ by Lemma 4.4.

This gives us a simple way to determine the $S_{\mu,\lambda}$ containing e. If (z_0, ζ_0) is the apex, we must have $|\mu z_0 - \lambda|^2 + \zeta_0^2 |\mu|^2 = 1$. Thus $|\mu| \leqslant \zeta_0^{-1}$. There are only a finite number of μ possible. For each one $|\mu z_0 - \lambda|^2 = 1 - \zeta_0^2 |\mu|^2$ which allows only a finite number of values for λ. We enumerate all solutions μ, λ, check whether $(\mu, \lambda) = \mathcal{O}$, and then whether $\lambda/\mu \in l$.

We can also save quite a bit of work by making use of some obvious symmetries [2]. Let π be the four group (elementary Abelian group of order 4) with generators c and e. Let π act on $\mathbf{C}$ by $c(z) = \bar{z}$, $e(z) = -z$. Let π act on H by $c(z, \zeta) = (c(z), \zeta)$, $e(z, \zeta) = (e(z), \zeta)$. Clearly $c(S_{\mu,\lambda}) = S_{\bar{\mu},\bar{\lambda}}$ and $e(S_{\mu,\lambda}) = S_{\mu,-\lambda}$. Therefore B is stable under π.

Let π act on $SL(2, \mathcal{O})$ as a group of automorphisms by $c(A) = \bar{A}$, the complex conjugate of A and $e(A) = EAE$, where $E = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathcal{O})$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $e(A) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. The action of π on the generators T, U, J is easy to determine. In fact $c(J) = J$, $c(T) = T$, $c(U) = T^t U^{-1}$, where t is the trace of ω, $e(J) = J$, $e(T) = T^{-1}$, $e(U) = U^{-1}$. If $A_i = \begin{pmatrix} \lambda' & \mu' \\ \mu & \lambda \end{pmatrix}$ is the generator corresponding to 2-cell $B \cap S_{\mu,\lambda}$, then $c(A_i) = \begin{pmatrix} \bar{\lambda}' & \bar{\mu}' \\ \bar{\mu} & \bar{\lambda} \end{pmatrix}$, so $B \cap c(A_i)^{-1}B = B \cap S_{\bar{\mu},\bar{\lambda}} = c(B \cap S_{\mu,\lambda})$, so $c(A_i) \in S$, and we can express $c(A_i)$ in terms of our chosen generators by writing $c(A_i) = [c(A_i)]$. Similarly, $B \cap e(A_i)^{-1}B = e(B \cap S_{\mu,\lambda})$, so $e(A_i) \in S$ and we can write $e(A_i) = [e(A_i)]$. Thus we can easily calculate the action of π on the generators of $SL(2, \mathcal{O})$.

Now if R is a formal word in the generators of $SL(2, \mathcal{O})$, define $c(R)$, $e(R)$, $ce(R)$ to be the formal words obtained by applying c, e, or ce to each generator occurring in R.

LEMMA 4.20. *If e is an edge of ∂B and $\sigma \in \pi$, we can choose $R_{\sigma(e)} = \sigma(R_e)$.*

Proof. Let S_{μ_i,λ_i} be the $S_{\mu,\lambda}$ containing e. Let $\alpha_i = \lambda_i/\mu_i$ be the center of S_{μ_i,λ_i}. Then $\sigma(S_{\mu_i,\lambda_i})$ is the $S_{\mu,\lambda}$ containing $\sigma(e)$ and its center is $\sigma(\alpha_i)$. If $\tau_i = \begin{pmatrix} \lambda_i' & \mu_i' \\ \mu_i & \lambda_i \end{pmatrix}$, a comparison of the definitions of the action of c and e on $\mathbf{C}$ and on $SL(2, \mathcal{O})$ shows that we can choose $\sigma(\tau_i)$ as the matrix corresponding to $\sigma(\alpha_i)$. The α_i and the $\sigma(\alpha_i)$ occur in the same order on their lines l and $\sigma(l)$. With this choice, $R_e = [\tau_1^{-1}][\tau_1\tau_2^{-1}] \cdots [\tau_s]$ and $R_{\sigma(e)} = [\sigma(\tau_1)^{-1}][\sigma(\tau_1)\, \sigma(\tau_2)^{-1}] \cdots [\sigma(\tau_s)]$. Let τ be any $\tau_i\tau_{i+1}^{-1}$. Let $[\tau] = J^\epsilon T^m U^n A_j T^p U^q$. Then

$$\sigma[\tau] = \sigma(J)^\epsilon\, \sigma(T)^m\, \sigma(U)^n\, [\sigma(A_j)]\, \sigma(T)^p\, \sigma(U)^q.$$

Using the relations (1) of Theorem 4.18, we can collect the terms in J, T, U and obtain the canonical form $[\sigma(\tau)]$. Therefore the relation $\sigma(R_e) = 1$ is equivalent to the relation $R_{\sigma(e)} = 1$ modulo the relations (1) of Theorem 4.18.

This result can save us quite a bit of matrix calculation. Instead of taking all $R_{e_i} = 1$ for a set of representatives of the edges mod $\mathcal{O}$, we take only the $R_{e_i} = 1$ for a set of representatives of the edges mod $\mathcal{O}$ *and* the action of π. We then transform the resulting relations by the elements of π to get all the edge relations.

It is also very easy to obtain a presentation for $GL(2, \mathcal{O})$ if we know the action of e on $SL(2, \mathcal{O})$. In fact, $GL(2, \mathcal{O})$ is the semidirect product of $SL(2, \mathcal{O})$ and the subgroup H consisting of all $\begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix}$, where η is a unit of $\mathcal{O}$. If $K \neq Q(\sqrt{-1}), Q(\sqrt{-3})$, then $H = \{1, E\}$, where $E = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ as above. Therefore we get a presentation of $GL(2, \mathcal{O})$ by adding the extra generator E and the extra relations $E^2 = 1$ and $EXE = e(X)$ for each generator X of $SL(2, \mathcal{O})$.

We can also obtain a presentation for the group $\widetilde{GL}(2, \mathcal{O})$ of all transformations of $\mathcal{O} \oplus \mathcal{O}$ which are either linear or antilinear. We need only add one new generator F and the new relations $F^2 = 1$, $FE = EF$, and $FXF = c(X)$ for each generator X of $SL(2, \mathcal{O})$. Here F is the transformation $F(\alpha, \beta) = (\bar{\alpha}, \bar{\beta})$. If $K = Q(\sqrt{-1})$ or $Q(\sqrt{-3})$, we must also include the effect of F on H.

5. The Case $Q(\sqrt{-1})$

For $K = Q(i)$ we have $\mathcal{O} = Z[i]$ and the units of $\mathcal{O}$ are ± 1, $\pm i$. Consider the $S_{\mu,\lambda}$ for $\mu = 1$. These are all equivalent mod $\mathcal{O}$. The hemisphere $S_{1,0}$ meets its neighbors $S_{1,\pm 1}$, $S_{1,\pm i}$ along the vertical planes given by $\xi = \pm\frac{1}{2}$, $\eta = \pm\frac{1}{2}$, where $Z = \xi + i\eta$. The lowest points of $S_{1,0}$ over the square $|\xi| \leqslant \frac{1}{2}$, $|\eta| \leqslant \frac{1}{2}$ occur over the corners and have $\zeta = 1/\sqrt{2}$. Therefore, the points B' lying above all $S_{1,\lambda}$ all have $\zeta \geqslant 1/\sqrt{2}$. Now the highest point on $S_{\mu,\lambda}$ has $\zeta = 1/|\mu|$. If $|\mu| > \sqrt{2}$, $S_{\mu,\lambda} \cap B' = \emptyset$. If $|\mu| = \sqrt{2}$, $S_{\mu,\lambda}$ meets B' in at most one point. If $|\mu| < \sqrt{2}$, then $\mu = \pm 1, \pm i$, so $S_{\mu,\lambda} = S_{1,\mu^{-1}\lambda}$. Thus the points of B' lie above all $S_{\mu,\lambda}$, so $B' = B$. The cell subdivision of ∂B projects onto the regular tesselation of $\mathbf{C}$ into squares with sides parallel to the axes, centers at the points $m + ni \in \mathcal{O}$, and sides of length 1.

All 2-cells of ∂B can be transformed into the cell $S_{0,1} \cap B$. Therefore beside the T, U, and L, we have only one generator $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The group Γ must be obtained from Lemma 4.12 and not Corollary 4.13. We choose $\omega = i$, so $U = \left(\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right)$. The subgroup of L_ρ is generated by $L = \left(\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}\right)$. It is convenient to keep J as a generator even though $L^2 = J$. The L_ρ are now $1, J, L, JL$. As always J is central and $J^2 = 1$. The relations (3) of Lemma 4.12 all follow from $L^{-1}TL = T^{-1}$, and $L^{-1}UL = U^{-1}$. The relation (4) is $A^{-1} = JA$. The relations (5) for $L_\rho = J$ follow from the fact that J is central. Also $AJL = JAL$ so we need only check the relation (5) for AL. An easy calculation shows $AL = JLA$.

We now must find the edge relations. Mòdulo $\mathcal{O}$ there are just two edges, one with $\xi = \frac{1}{2}$ and center $\frac{1}{2}$, and one with $\eta = \frac{1}{2}$ and center $i/2$. The first lies only on $S_{1,0}$ and $S_{1,1}$ while the second lies only on $S_{1,0}$ and

$S_{1,i}$. For the first we take $\tau_1 = A$, $\tau_2 = \binom{0\ -1}{1\ \ \ 1} = AT$ and get the relation $[A^{-1}][AT^{-1}A^{-1}][AT] = 1$. Now $A^{-1} = JA$, AT is in its canonical form, and $AT^{-1}A^{-1} = \binom{1\ 0}{1\ 1} = TAT$, so $[AT^{-1}A^{-1}] = TAT$. Therefore our relation is $JA \cdot TAT \cdot AT = 1$ or $(AT)^3 = J$. For the second edge $\alpha_1 = 0/1$, $\alpha_2 = i/1$ and we take $\tau_1 = A$, $\tau_2 = \binom{0\ -1}{1\ \ \ i} = AU$. The relation is $[A^{-1}][AU^{-1}A^{-1}][AU] = 1$. Now $AU^{-1}A^{-1} = \binom{1\ 0}{i\ 1} = L\binom{i\ \ \ 0}{1\ -i} = LUAU^{-1}$ so our relation is

$$A \cdot JLUAU^{-1} \cdot AU = 1 \qquad \text{or} \qquad ALUAU^{-1}AU = J$$

or

$$UALU = JA^{-1}UA^{-1} = JAUA.$$

THEOREM 5.1. *If $K = Q(\sqrt{-1})$, $\mathcal{O} = Z[i]$, the group $SL(2, \mathcal{O})$ is generated by the elements*

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with the relations

$$TU = UT, \quad J^2 = 1, \quad J \text{ central}, \quad L^2 = J, \quad (TL)^2 = J,$$
$$(UL)^2 = J, \quad (AL)^2 = J, \quad A^2 = J, \quad (TA)^3 = J, \quad (UAL)^3 = J.$$

The last relation is equivalent to $UALU = JAUA$ modulo the other relations.

In view of what we have done above, only the last statement needs to be verified. If $UALU = JAUA$, then $UALUAL = JAUJL = AUL$ but $(UL)^2 = J$, so

$$(UAL)^3 = AUL \cdot UAL = A(UL)^2L^{-1}AL = JAL^{-1}AL = ALAL = J.$$

Performing these steps backwards we easily verify the converse.

COROLLARY 5.2. *If $K = Q(\sqrt{-1})$, then*

$$SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})] = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$$

with generators U and L.

We can easily obtain a presentation for $GL(2, \mathcal{O})$. It is only necessary to add a new generator $W = \binom{i\ 0}{0\ 1}$ and the new relations $W^4 = 1$,

$WJ = JW, \quad WTW^{-1} = U, \quad WUW^{-1} = T^{-1}, \quad WAW^{-1} = JLA,$
$WL = LW.$

COROLLARY 5.3. *If* $K = Q(\sqrt{-1})$, *then*

$$GL(2, \mathcal{O})/[GL(2, \mathcal{O}), GL(2, \mathcal{O})] = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z}$$

generated by U *and* W.

To get $\widetilde{GL}(2, \mathcal{O})$ we add a new generator F with the relations $F^2 = 1$, $FWF = W^{-1}, FLF = L^{-1}, FUF = U^{-1}, FTF = T, FJF = J, FAF = A$.

6. THE CASE $Q(\sqrt{-3})$

For $K = Q(\sqrt{-3})$, $\mathcal{O} = Z[\omega]$, where $\omega = \frac{1}{2}(-1 + \sqrt{-3})$ is a primitive cube root of 1. Consider the $S_{1,\lambda}$, $\lambda \in \mathcal{O}$ and let B' be the set of points lying above all $S_{1,\lambda}$. Again $S_{1,0}$ and $S_{1,1}$ meet in an arc e of a circle with center $1/2$ in the vertical plane $\xi = 1/2$. Since the lattice formed by $\mathcal{O}$ has hexagonal symmetry, the remaining intersections of $S_{1,0}$ with its neighbors are obtained from e by rotations about 0 through multiples of $\pi/3$. Thus the part of $S_{1,0}$ not covered by the remaining $S_{1,\lambda}$ lie over a regular hexagonal region with center 0 and with two of its sides lying on the lines $\xi = \pm 1/2$. The lowest points of B' on $S_{1,0}$ lie over the corners of this hexagon. These corners have distance $1/\sqrt{3}$ from 0, so the smallest value for ζ in B' is $\zeta = \sqrt{2/3}$. If $S_{\mu,\lambda}$ meets B', then $1/|\mu| \geqslant \sqrt{2/3}$ or $|\mu|^2 = N\mu \leqslant 3/2$ so $N\mu = 1$ and μ is a unit of $\mathcal{O}$. Therefore $S_{\mu,\lambda} = S_{1,\mu^{-1}\lambda}$. Thus shows that all $S_{\mu,\lambda}$ lie below B' so $B' = B$.

There is only one 2-cell mod $\mathcal{O}$ so the 2-cells contribute one generator $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For the remaining generators we have $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $L = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$. The subgroup consisting of the L_ρ is defined by $J^2 = 1, L^3 = 1$, $JL = LJ$ because the only units of $\mathcal{O}$ are ± 1, $\pm\omega$, $\pm\omega^2$. J is central. The action of L on T and U is given by $L^{-1}TL = T^{-1}U^{-1}$, $L^{-1}UL = T$ using $\omega^2 + \omega + 1 = 0$. Also $A^{-1} = JA$. We must examine the relations (5) of Lemma 4.12. Since J is central it will suffice to look at AL and AL^2. Now $AL = L^2A$. This implies $AL^2 = L^4A = LA$.

We must now determine the edge relations. The cell subdivision of ∂B projects onto the regular hexagonal tesselation of $\mathbf{C}$ into hexagons with centers at the points of $\mathcal{O}$ and sides of length $1/\sqrt{3}$. There are 3 edges mod $\mathcal{O}$, say, e_1, e_2, e_3, where e_1 lies on $S_{1,0}$ and $S_{1,1}$, e_2 lies on $S_{1,0}$

and $S_{1,-\omega}$, and e_3 lies on $S_{1,0}$ and $S_{1,\omega}$. For e_1 we have $\alpha_1 = 0/1$, $\alpha_2 = 1/1$. Choose $\tau_1 = A$, $\tau_2 = AT$. As in § 5, this yields the relations $(AT)^3 = J$. For e_2, $\alpha_1 = 0/1$, $\alpha_2 = -\omega/1$. Choose $\tau_1 = A$, $\tau_2 = \left(\begin{smallmatrix}0 & -1\\ 1 & -\omega\end{smallmatrix}\right) = AU^{-1}$. This gives $[A^{-1}][AUA^{-1}][AU^{-1}] = 1$. Now

$$AUA^{-1} = \begin{pmatrix}1 & 0\\ -\omega & 1\end{pmatrix} = JL\begin{pmatrix}-\omega & 0\\ 1 & -\omega^2\end{pmatrix} = JL\begin{pmatrix}-\omega & 0\\ 1 & \omega+1\end{pmatrix} = JLU^{-1}AUT,$$

so our relation is $JA \cdot JLU^{-1}AUT \cdot AU^{-1} = 1$ or $ALU^{-1}AUTAU^{-1} = 1$. For e_3, $\alpha_1 = 0/1$, $\alpha_2 = \omega/1$. Choose $\tau_1 = A$, $\tau_2 = \left(\begin{smallmatrix}0 & -1\\ 1 & \omega\end{smallmatrix}\right) = AU$, getting $[A^{-1}][AU^{-1}A^{-1}][AU] = 1$. But $AU^{-1}A^{-1} = JT^{-1}U^{-1}A^{-1}UL^{-1}$ so we get

$$JA \cdot JT^{-1}U^{-1}A^{-1}UL^{-1} \cdot AU = 1 \qquad \text{or} \qquad AT^{-1}U^{-1}A^{-1}UL^{-1}AU = 1.$$

The inverse of this is $U^{-1}ALU^{-1}AUTA = 1$ using $A^2 = J$. But this is equivalent to the relation obtained from e_2. We can rewrite this as $AUT = UL^2AUA = UALUA$.

THEOREM 6.1. *Let $K = Q(\sqrt{-3})$ and let $\mathcal{O} = Z[\omega]$, where $\omega = \frac{1}{2}(-1+\sqrt{-3})$, be the ring of integers of K. Then $SL(2,\mathcal{O})$ is generated by the elements*

$$T = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\quad U = \begin{pmatrix}1 & \omega\\ 0 & 1\end{pmatrix},\quad J = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},\quad L = \begin{pmatrix}\omega^2 & 0\\ 0 & \omega\end{pmatrix},\quad A = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

with the relations

$$TU = UT,\qquad J^2 = 1,\qquad J \text{ central},\qquad L^3 = 1,\qquad L^{-1}TL = T^{-1}U^{-1},$$

$$L^{-1}UL = T,\qquad A^2 = J,\qquad (AL)^2 = J,\qquad (TA)^3 = J,\quad (UAL)^3 = J.$$

The last relation is equivalent to $UALUA = AUT$ modulo the remaining relations.

Again only the last statement needs verification. If $UALUA = AUT$, then

$$(UAL)^2 = AUTL$$

or

$$(UAL)^3 = AUTLUAL = AUT \cdot T^{-1}U^{-1}L \cdot AL = ALAL = J$$

using $LU = T^{-1}U^{-1}L$. This argument reverses.

COROLLARY 6.2. *If* $K = Q(\sqrt{-3})$, *then*

$$SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})] = \mathbf{Z}/3\mathbf{Z}$$

generated by U.

To obtain a presentation for $GL(2, \mathcal{O})$, we need only add one new generator $W = \begin{pmatrix} -\omega & 0 \\ 0 & 1 \end{pmatrix}$ and the new relations $W^6 = 1$, $WTW^{-1} = U^{-1}$, $WUW^{-1} = TU$, $WAW^{-1} = JL^2A$, $WJ = JW$, $WL = LW$.

COROLLARY 6.3. *If* $K = Q(\sqrt{-3})$,

$$GL(2, \mathcal{O})/[GL(2, \mathcal{O}), GL(2, \mathcal{O})] = \mathbf{Z}/6\mathbf{Z}$$

generated by W.

To get $\widetilde{GL}(2, \mathcal{O})$ we add a new generator F with the relations $F^2 = 1$, $FWF = W^{-1}$, $FLF = L^{-1}$, $FUF = U^{-1}T^{-1}$, $FTF = T$, $FJF = J$, $FAF = A$.

7. SINGULAR POINTS

We must now discuss methods for finding the region B and the cell structure of ∂B. Our first problem is clearly that of finding all the singular points. This was partly done in Proposition 3.11.

DEFINITION. Let $\mathfrak{A}$ be any ideal of $\mathcal{O}$. We say an element $\beta \in \mathfrak{A}$ is *minimal* if $|\beta|$ is minimal among all $|\alpha|$ for $\alpha \in \mathfrak{A}$, $\alpha \neq 0$.

Since $\mathfrak{A}$ is a lattice in $\mathbf{C}$, there are only a finite number of minimal elements in $\mathfrak{A}$.

PROPOSITION 7.1. *Let* $\mathfrak{A}_0 = \mathcal{O}$, $\mathfrak{A}_1, \ldots, \mathfrak{A}_{h-1}$ *represent the* h *ideal classes of* $\mathcal{O}$. *For each* $i = 1, \ldots, h-1$, *write* $\mathfrak{A}_i = (\beta, \alpha)$ *in all possible ways with* β *minimal in* $\mathfrak{A}_i$ *and form* α/β. *In this way we get all possible singular points, each exactly twice, once as* α/β *and once as* $-\alpha/-\beta$.

Proof. We have seen in proving Proposition 3.11 that every singular point is obtained in this way and that α/β is not singular if β is not minimal in (α, β). Now if $(\alpha, \beta) = \mathfrak{A}_0$, then $|\beta \cdot \alpha/\beta - \alpha| = 0 < 1$ and $(\beta, \alpha) = \mathcal{O}$, so α/β is not singular. In all the other cases, however, we do get a singular point. In fact, if $(\mu, \lambda) = \mathcal{O}$, then $|\mu(\alpha/\beta) - \lambda| = |\mu\alpha - \lambda\beta| \, |\beta|^{-1} \geqslant 1$ because $\mu\alpha - \lambda\beta \in \mathfrak{A}_i$ so by the minimality of β, $|\mu\alpha - \lambda\beta| \geqslant |\beta|$. Note that $\mu\alpha - \lambda\beta = 0$ is impossible since it implies $\alpha = u\lambda$, $\beta = u\mu$ for some $u \in K$ and so $\mathfrak{A}_i = (\alpha, \beta) = u(\lambda, \mu) = \mathcal{O}u$, but we are assuming $i \neq 0$, so $\mathfrak{A}_i$ is not principal.

We must now check the uniqueness. Suppose $\alpha/\beta = \gamma/\delta$, $(\beta, \alpha) = \mathfrak{A}_i$, $(\delta, \gamma) = \mathfrak{A}_j$ with $i, j \neq 0$, β, δ minimal in $\mathfrak{A}_i$, $\mathfrak{A}_j$, respectively. Then for some $u \in K$ we have $\gamma = u\alpha$, $\beta = u\delta$. Therefore $\mathfrak{A}_i = u\mathfrak{A}_j$ but $\mathfrak{A}_i$, $\mathfrak{A}_j$ are not equivalent unless $i = j$. Thus $i = j$ and $\mathfrak{A}_i = u\mathfrak{A}_j$, so u is a unit of $\mathcal{O}$. This implies $u = \pm 1$ unless $K = Q(\sqrt{-1})$ or $Q(\sqrt{-3})$. But these exceptional fields have $h = 1$.

This result gives us a method to enumerate all singular points by enumerating all ideal classes and determining the minimal elements. Note that for each $i \neq 0$ and minimal $\beta \in \mathfrak{A}_i$, the singular points of the form α/β for $\mathfrak{A}_i = (\beta, \alpha)$ form a finite number of cosets $\alpha/\beta + \mathcal{O}$ of $\mathcal{O}$ in K. This is clear because $(\beta, \alpha) = (\beta, \alpha + \nu\beta)$ for all ν. To count the number of these cosets, we must find the number of values of α modulo β such that $(\beta, \alpha) = \mathfrak{A}_i$. Now the assertion $(\beta, \alpha) = \mathfrak{A}_i$ is equivalent to the assertion that α modulo β generates the finite $\mathcal{O}$-module $\mathfrak{A}_i/(\beta)$. Since β is minimal in $\mathfrak{A}_i$, $\mathbf{Z}\beta$ is a pure subgroup of $\mathfrak{A}_i$ so $\mathfrak{A}_i/\mathbf{Z}\beta$ is infinite cyclic. Thus $\mathfrak{A}_i/(\beta)$ is a finite cyclic group of order $m = N\mathfrak{A}_i N(\beta)^{-1}$. This value is independent of the choice of β since all minimal β have the same value for $N(\beta) = |\beta|^2$. Now every subgroup of a cyclic group is stable under all endomorphisms, so an element $\bar{\alpha} \in \mathfrak{A}_i/(\beta)$ generates this as an $\mathcal{O}$-module if and only if it generates it as an Abelian group. The number of such generators is $\varphi(m)$ where φ is the Euler φ function.

In any ideal the number of minimal elements is even since $-\beta$ is minimal if β is.

Definition. If $\mathfrak{A}$ is an ideal of $\mathcal{O}$, let $2e(\mathfrak{A})$ be the number of minimal elements of $\mathfrak{A}$. Let $h(\mathfrak{A})$ be the index $[\mathfrak{A} : (\beta)]$ where β is any minimal element of $\mathfrak{A}$. These numbers clearly depend only on the class of $\mathfrak{A}$.

Corollary 7.2. [10]: *Let $\mathfrak{A}_0 = \mathcal{O}$, $\mathfrak{A}_1, \ldots, \mathfrak{A}_{h-1}$ represent the h ideal classes of $\mathcal{O}$. Let $e_i = e(\mathfrak{A}_i)$ and $h_i = h(\mathfrak{A}_i)$. Then the singular points form a finite union of cosets of $\mathcal{O}$, the number of these cosets being $\sum_1^{h-1} e_i\varphi(h_i)$.*

We now give an even more explicit construction for the singular points by elaborating on the methods of Humbert [10]. See also [4, Chap. 2, Section 7.7].

Let $K = Q(\sqrt{-m})$ with $m > 0$, squarefree, and $m \neq 3$. Let $\omega = \sqrt{-m}$ if $m \not\equiv 3 \bmod 4$ and $\omega = \frac{1}{2}(-1 + \sqrt{-m})$ if $m \equiv 3 \bmod 4$.

DEFINITION. Let I be the set of all pairs (q, a) with $q, a \in \mathbf{Z}$, $q > 0$, $-q/2 < a \leqslant q/2$, $q^2 \leqslant N(a + \omega)$ and $q \mid N(a + \omega)$. For $(q, a) \in I$, let $\mathfrak{A}_{q,a} = (q, a + \omega)$.

THEOREM 7.3. *Every ideal class is represented by some $\mathfrak{A}_{q,a}$ for $(q, a) \in I$. This correspondence between I and the ideal class group is one-to-one except for the identification $\mathfrak{A}_{q,a} \approx \mathfrak{A}_{q,b}$ if $q^2 = N(a + \omega)$ and $b = -a$ for $m \not\equiv 3 \bmod 4$, $b = 1 - a$ for $m \equiv 3 \bmod 4$.*

The minimal elements of $\mathfrak{A}_{q,a}$ are just $\pm q$ unless $q^2 = N(a + \omega)$ when they are $\pm q$ and $\pm(a + \omega)$.

If $q^2 < N(a + \omega)$, $\mathfrak{A}_{q,a}$ is the unique ideal of least norm in its class. If $q^2 = N(a + \omega)$, there are just two ideals of least norm in the class of $\mathfrak{A}_{q,a}$, viz., $\mathfrak{A}_{q,a}$ and $\mathfrak{A}_{q,b}$, where b is as above.

Note that if b is defined as above, then $\overline{a + \omega} = -(b + \omega)$. Therefore $\bar{\mathfrak{A}}_{q,a} = \mathfrak{A}_{q,b}$. Also $(q, b) \in I$ unless $a = q/2$ or $(q + 1)/2$. In this case we have $(q, b - q) \in I$. Clearly, $\mathfrak{A}_{q,b} = \mathfrak{A}_{q,b-q}$.

In the course of proving this theorem we will also show that $q, a + \omega$ is a base for $\mathfrak{A}_{q,a}$ over $\mathbf{Z}$ and that $\mathcal{O}/\mathfrak{A}_{q,a}$ is a cyclic group of order q. We will also see that for $(q, a) \in I$ we have $q \leqslant \sqrt{|\Delta|/3}$ where Δ is the discriminant. This reproves a theorem of M. Kneser [9, p. 544] to the effect that in each ideal class of a quadratic imaginary field there is an ideal of norm $\leqslant \sqrt{|\Delta|/3}$.

COROLLARY 7.4. *A complete set of representatives for the singular points of K mod $\mathcal{O}$ is given by the set of all $p(a + \omega)/q$, where $(q, a) \in I$, $q \neq 1$, and p runs over a set of representatives, modulo q of the integers (in $\mathbf{Z}$) relatively prime to q.*

Proof. Let $\mathfrak{A} \subset \mathcal{O}$ be any ideal. Of all $c + d\omega \in \mathfrak{A}$ with $d \neq 0$, choose one with least d. Then, if $x + y\omega \in \mathfrak{A}$, we have $d \mid y$ so $x + y\omega = yd^{-1}(c + d\omega) + z$ for $z \in \mathbf{Z}$. Let $f > 0$ be the least positive integer in $\mathfrak{A}$. Then $\mathfrak{A} = \mathbf{Z} \cdot (c + d\omega) + \mathbf{Z} \cdot f$. Now $f\omega \in \mathfrak{A}$, so $d \mid f$. Also $\omega(c + d\omega) \in \mathfrak{A}$. If $m \not\equiv 3$ (4), $\omega(c + d\omega) = c\omega - md$. If $m \equiv 3$ (4), $\omega(c + d\omega) = c\omega - d(\omega + (m + 1)/4)$. Therefore we have $d \mid c$ or $d \mid c - d$ so $d \mid c$ in all cases. Let $c = ad, f = qd$. Then $\mathfrak{A} = d(q, a + \omega)$ and $(q, a + \omega) = \mathbf{Z}q + \mathbf{Z}(a + \omega)$. Since

$$N(a + \omega) = (a + \bar{\omega})(a + \omega) \in \mathfrak{A} \cap \mathbf{Z},$$

the definition of q shows that $q \mid N(a + \omega)$. Note that whenever $q, a \in \mathbf{Z}$ and $q \mid N(a + \omega)$ we have $(q, a + \omega) = \mathbf{Z}q + \mathbf{Z}(a + \omega)$. It will suffice to show that the right side is an ideal. But $\omega \cdot q = q(a + \omega) - aq$ and to show $\omega(a + \omega)$ lies in it, it will suffice to look at $(a + \bar{\omega}) \cdot (a + \omega) = N(a + \omega)$.

Now suppose $q \mid N(a + \omega)$. Then $\mathcal{O}/(q, a + \omega)$ is cyclic of order q generated by the image of $1 \in \mathcal{O}$. This is trivial to verify since $(q, a + \omega)$ is generated by q and $a + \omega$ as an Abelian group. We can clearly reduce a modulo q and assume $-q/2 < a \leqslant q/2$.

Now let $\mathfrak{A}$ have least norm in its class. Write $\mathfrak{A} = d(q, a + \omega)$ as above. Since $N\mathfrak{A}$ is least, $d = 1$. Suppose that $q^2 > N(a + \omega)$. Then

$$\mathfrak{A} \sim \mathfrak{A}_1 = \frac{a + \bar{\omega}}{q}\,\mathfrak{A} = \left(a + \bar{\omega}, \frac{N(a + \omega)}{q}\right) \quad \text{but} \quad q_1 = \frac{N(a + \omega)}{q} < q.$$

Since $\bar{\omega}$ is ω or $-1 - \omega$, $\mathfrak{A}_1 = (q_1, a_1 + \omega)$ where $a_1 + \omega = -(a + \bar{\omega})$ and $a_1 = -a$ or $1 - a$. Note $q_1 \mid N(a_1 + \omega)$ since $N(a_1 + \omega) = N(a + \omega) = qq_1$. Now $N\mathfrak{A} = q > N\mathfrak{A}_1 = q_1$ contradicting the minimality of $N\mathfrak{A}$. Therefore if $\mathfrak{A}$ has least norm in its class, then $\mathfrak{A} = \mathfrak{A}_{q,a}$ for some $(q, a) \in I$. When $q^2 = N(a + \omega)$ the argument just given shows $\mathfrak{A} \sim \mathfrak{A}_1 = (q, a_1 + \omega) = (q, a + \bar{\omega})$, where $a_1 = -a$ or $1 - a$ for $\bar{\omega} = -\omega$ or $-1 - \omega$ and $N\mathfrak{A}_1 = q = N\mathfrak{A}$. Clearly $\mathfrak{A}_1 = \bar{\mathfrak{A}}$. Note that $(q, a_1) \in I$. To see this suppose first that $m \not\equiv 3 \bmod 4$, so $a_1 = -a$. The only bad case is $a = q/2$. But $q^2 = N(a + \omega)$ or $q^2 = a^2 + m$. If $a = q/2$, $m = 3a^2$ so $m = 3$ since m is squarefree. But $m \not\equiv 3 \bmod 4$, so this case cannot occur. If $m \equiv 3 \bmod 4$, we have $a_1 = 1 - a$ or $a_1 - \frac{1}{2} = -(a - \frac{1}{2})$.

Now $-q/2 < a \leqslant q/2$ or $-q < 2a \leqslant q$. Since these are integers this is equivalent to $-q + 1 \leqslant 2a < q + 1$ or $-q/2 \leqslant a - \frac{1}{2} < q/2$. The only bad case is $a - \frac{1}{2} = -q/2$. Now $N(a + \omega) = (a - \frac{1}{2})^2 + m/4$. If $q^2 = N(a + \omega)$ and $a - \frac{1}{2} = -q/2$, we have $q^2 = q^2/4 + m/4$, so $m = 3q^2$. Since m is squarefree, $m = 3$.

If $m \not\equiv 3 \bmod 4$, we have $q^2 \leqslant N(a + \omega) = a^2 + m \leqslant \frac{1}{4}q^2 + m$, so $q \leqslant \sqrt{4m/3}$. If $m \equiv 3 \bmod 4$, we have $q^2 \leqslant N(a + \omega) = (a - \frac{1}{2})^2 + m/4 \leqslant \frac{1}{4}q + \frac{1}{4}m$, so $q \leqslant \sqrt{m/3}$. Thus, for all $(q, a) \in I$, $q \leqslant \sqrt{|\Delta|/3}$ where Δ is the discriminant of K.

We now determine the minimal elements of $\mathfrak{A}_{q,a}$ using the bound just obtained. Let $\alpha = xq + y(a + \omega)$ be any element of $\mathfrak{A}_{q,a}$ with $x, y \in \mathbf{Z}$. For $m \not\equiv 3 \bmod 4$, $|\alpha|^2 = (xq + ya)^2 + y^2 m$ but $q^2 \leqslant 4m/3$. Thus $|\alpha|^2 > q^2$ unless $y = 0, \pm 1$. If $m \equiv 3 \bmod 4$,

$$|\alpha|^2 = (xq + ya - \tfrac{1}{2}y)^2 + y^2 \frac{m}{4}$$

but $q^2 \leqslant m/3 < |\alpha|^2$ unless $y = 0, \pm 1$. For $y = 0$, the only candidates for minimal elements are $\pm q$. For $y = 1$, $\alpha = xq + a + \omega$. For $m \not\equiv 3 \bmod 4$, $|\alpha|^2 = (xq + a)^2 + m$. We must minimize $xq + a$, i.e., choose x so $|xq + a| \leqslant q/2$. The only possibility is $x = 0$ unless $a = q/2$. As we observed above, we can only have $a = q/2$ when $q^2 < N(a + \omega)$, in which case $q^2 < (xq + a)^2 + m$ for $x = 0, -1$. In the case $m \equiv 3 \bmod 4$, $|\alpha|^2 = (xq + a - \frac{1}{2})^2 + m/4$. Again we want $|xq + a - \frac{1}{2}| \leqslant q/2$, so $x = 0$ unless $a - \frac{1}{2} = -q/2$. As we observed above, this implies $q^2 < N(a + \omega)$ unless $m = 3$. For the case $y = -1$ we need only consider $-\alpha$. Therefore the only candidates for minimal elements are $\pm q$, $\pm(a + \omega)$. These are all minimal if $q^2 = N(a + \omega)$. Otherwise only $\pm q$ are minimal since $q^2 \leqslant N(a + \omega)$.

To finish the proof of Theorem 7.3 we must show that all $\mathfrak{A}_{q,a}$ lie in different ideal classes except for the noted identification. The last statement of Theorem 7.3 then follows, since the ideals of least norm in any class are of the form $\mathfrak{A}_{q,a}$ for $(q, a) \in I$. To see that $\mathfrak{A}_{q,a} \neq \mathfrak{A}_{q,a_1}$, note that $\mathfrak{A}_{q,a_1} = (a - \omega)/q\mathfrak{A}_{q,a}$. If the ideals are equal, then $(a - \omega)/q$ is a unit of $\mathcal{O}$. This is impossible unless $m = 1$ or 3. For $m = 1$, $q \leqslant \sqrt{4m/3}$ implies $q = 1$ so $a = 0 = a_1$. The case $m = 3$ was specifically excluded.

To see that the $\mathfrak{A}_{q,a}$ lie in different classes, it will suffice to prove Corollary 7.4 and check that the various $\mathfrak{A}_{q,a}$ lead to singular points not congruent mod $\mathcal{O}$. We then apply Proposition 7.1.

We have already noted that q is minimal in $\mathfrak{A}_{q,a}$. Now $\mathfrak{A}_{q,a}/(q)$ is a cyclic group of order q generated by the image of $a + \omega$. To see this note $\mathfrak{A}_{q,a} = \mathbf{Z} \cdot q + \mathbf{Z} \cdot (a + \omega)$ and $(q) = \mathbf{Z}q + \mathbf{Z}q\omega$ but $q\omega = q \cdot (a + \omega) - a \cdot q$. Therefore $\mathfrak{A}_{q,a}/(q) = \mathbf{Z} \cdot (a + \omega)/\mathbf{Z} \cdot q(a + \omega)$. As we observed in proving Corollary 7.2, we can get a set of representatives for the singular points coming from $\mathfrak{A}_{q,a}$ with denominator q by choosing a set of representatives α for the generators of $\mathfrak{A}_{q,a}/(q)$. As these representatives, we choose $p(a + \omega)$ where p runs over a set of representatives modulo q for the integers relatively prime to q. In the case $q^2 = N(a + \omega)$, we must take only one of the two ideals $\mathfrak{A}_{q,a}$ and $\mathfrak{A}_{q,a_1}$. But $\mathfrak{A}_{q,a}$ also has $a + \omega$ as minimal element. The isomorphism $\mathfrak{A}_{q,a} \approx \mathfrak{A}_{q,a_1}$, considered above, sends $x \in \mathfrak{A}_{q,a}$ to $((a + \bar{\omega})/q)x$. In particular, it sends $a + \omega$ to $N(a + \omega)/q = q$. Thus instead of considering the minimal element $a + \omega$ of $\mathfrak{A}_{q,a}$ we can consider the minimal element q of $\mathfrak{A}_{q,a_1}$.

We must now check that the points obtained are all distinct mod $\mathcal{O}$. Suppose $p(a + \omega)/q$ and $p'(a' + \omega)/q'$ are two of the points obtained. If

$p'(a' + \omega)/q' = p(a + \omega)/q + x + y\omega;\ x, y \in \mathbf{Z}$, then $p'/q' \equiv p/q \bmod 1$. Since $(p, q) = 1 = (p', q')$, we have $q' = q$ and $p' \equiv p \bmod q$. Also $p'a'/q' \equiv pa/q \bmod 1$, so $p'a' \equiv pa \bmod q$ or $pa' \equiv pa \bmod q$. But $(p, q) = 1$ so $a' \equiv a \bmod q$. Therefore $a' = a$, $p' = p$.

Finally, we have $\mathfrak{A}_{1,0} = \mathcal{O}$ and no other $\mathfrak{A}_{q,a}$ is equivalent to this. This concludes the proof of Theorem 7.3.

We now make the conditions defining I a bit more explicit.

LEMMA 7.5. (1) *If* $m \not\equiv 3 \bmod 4$, *I consists of all* (s, r) *satisfying* $r, s \in \mathbf{Z}$, $s > 0$, $-s/2 < r \leqslant s/2$, $s \mid r^2 + m$ *and* $s^2 \leqslant r^2 + m$. *Also* $\mathfrak{A}_{s,r} = (s, r + \sqrt{-m})$.

(2) *If* $m \equiv 3 \bmod 4$, *I consists of all* $(q, a) = (s/2, (r - 1)/2)$ *where* $r, s \in \mathbf{Z}$, $s > 0$, *s even*, $-s/2 \leqslant r < s/2$, $2s \mid r^2 + m$, *and* $s^2 \leqslant r^2 + m$. *Also* $\mathfrak{A}_{q,a} = \frac{1}{2}(s, r + \sqrt{-m})$.

Proof. In case (1) we have just written s, r for q, a. In case (2), $a + \omega = a - \frac{1}{2} + \frac{1}{2}\sqrt{-m}$, so $2\mathfrak{A}_{q,a} = (2q, 2a - 1 + \sqrt{-m})$. Let $s = 2q$, $r = 2a - 1$. Then $a + \omega = \frac{1}{2}(r + \sqrt{-m})$, so $N(a + \omega) = (r^2 + m)/4$. Thus $q \mid N(a + \omega)$ if and only if $2s \mid r^2 + m$ and $q^2 \leqslant N(a + \omega)$ if and only if $s^2 \leqslant r^2 + m$. Finally, $-q/2 < a \leqslant q/2$ if and only if $-2q < 4a \leqslant 2q$, i.e., $-s < 2r + 2 \leqslant s$. Since s is even, this is equivalent to $-s \leqslant 2r < s$. Note that $2s \mid r^2 + m$ implies that r is odd, so we can solve $2a - 1 = r$ for a.

COROLLARY 7.6. *The singular points of K,* $\bmod \mathcal{O}$, *are given by* $p(r + \sqrt{-m})/s$, *where* $r, s \in \mathbf{Z}$, $s > 0$, $-s/2 < r \leqslant s/2$, $s^2 \leqslant r^2 + m$, *and*

(1) *If* $m \not\equiv 3 \bmod 4$, $s \mid r^2 + m$, $s \neq 1$, $(p, s) = 1$ *and p is taken* mod s,

(2) *If* $m \equiv 3 \bmod 4$, *s is even*, $s \neq 2$, $2s \mid r^2 + m$, $(p, s/2) = 1$, *and p is taken* mod $s/2$.

The only thing to verify here is that changing the condition $-s/2 \leqslant r < s/2$ to $-s/2 < r \leqslant s/2$ in case (2) does not affect the result. If $r = -s/2$, then $s/2 = r + s$ and $p(r + s - \sqrt{-m})/s \equiv p(r - \sqrt{-m})/s \bmod \mathcal{O}$.

COROLLARY 7.7. *If* $m = 3$, $e(\mathfrak{A}) = 3$ *for all* $\mathfrak{A}$. *If m is even*, $e(\mathfrak{A}) = 1$ *for all* $\mathfrak{A}$. *If m is odd*, $m \neq 3$, *then* $e(\mathfrak{A}) = 1$ *or* 2 *for all* $\mathfrak{A}$ *and the number of ideal classes with* $e(\mathfrak{A}) = 2$ *is equal to the number of integers d such that* $d \mid m$ *and* $\sqrt{m/3} \leqslant d \leqslant \sqrt{m}$.

Proof. The cases $m = 1, 3$ are obvious. By Theorem 7.3, $e(\mathfrak{A}) = 1$ or 2 for all $\mathfrak{A}$ unless $m = 3$. The number of classes with $e(\mathfrak{A}) = 2$ is equal to half the number of pairs (r, s) with $r, s \in \mathbf{Z}$, $s > 0$, $-s/2 < r \leqslant s/2$ and $s^2 = r^2 + m$. The last equation is equivalent to $q^2 = N(a + \omega)$. The remaining conditions follow from those already given. Clearly, $s \mid r^2 + m$. If $m \equiv 3 \bmod 4$, the equation $s^2 = r^2 + m$ taken mod 4 implies that s is even and so $2s \mid r^2 + m$.

If m is even, then $m \equiv 2 \bmod 4$ since m is squarefree. Therefore, the equation $s^2 = r^2 + m$ cannot be satisfied mod 4. Thus $e(\mathfrak{A}) = 1$ for all $\mathfrak{A}$. We assume now that m is odd.

Now $r \neq 0$ since $m \neq s^2$. Therefore, we can find $e(\mathfrak{A})$ by counting the number of solutions with $r > 0$. Note that if $s = 1$ or 2, then $m = s^2 - r^2 \leqslant 4 - 1 = 3$ so we may assume all solutions have $s \geqslant 3$.

Let $d = s - r$, $d' = s + r$. Then $dd' = m$, $d, d' > 0$, $2s = d + d'$, and $2r = d' - d$. Since $2r \leqslant s$, we have $2d' - 2d \leqslant d + d'$ or $d' \leqslant 3d$. Therefore $m = dd' \leqslant 3d^2$ or $d \geqslant \sqrt{m/3}$. Also $d' > d$ so $d \leqslant \sqrt{m}$.

Conversely, given $d \mid m$, $\sqrt{m/3} \leqslant d \leqslant \sqrt{m}$, let $d' = m/d$. Then $d' > d$, $d' < 3d$. Since m is odd, so are d and d'. Let $s = \frac{1}{2}(d + d')$, $r = \frac{1}{2}(d' - d)$. Then $s^2 = r^2 + m$, $s, r > 0$, and $r \leqslant s/2$.

8. Determination of B

We will now discuss methods for finding the region B and the cell structure of ∂B. Suppose we are given a set of elements $\alpha_i = \lambda_i/\mu_i \in K$ with $(\lambda_i, \mu_i) = \mathcal{O}$, $i = 1,\ldots, n$. Let $S(\alpha_i) = S_{\mu_i,\lambda_i}$ for convenience and define $B(\alpha_1, \ldots, \alpha_n)$ to be the set of all $(z, \zeta) \in H$ lying above or on all $S(\alpha_i + \gamma)$, $i = 1,\ldots, n$, $\gamma \in \mathcal{O}$. I will give a method for deciding when $B(\alpha_1, \ldots, \alpha_n) = B$ and also for finding the cell decomposition of ∂B. This gives us an effective way to find B by adding more and more elements to the set $\{\alpha_1, \ldots, \alpha_n\}$ until we find $B(\alpha_1, \ldots, \alpha_n) = B$. We will discuss later the problem of giving an a priori bound for the number of steps needed to reach B. In the specific cases to be considered later, the required $\alpha_1, \ldots, \alpha_n$ may be found in Bianchi's paper [2]. Our method may then be applied to check his results and to find the cell structure of ∂B.

I will say that $S_{\beta,\alpha}$ *covers* $S_{\mu,\lambda}$ at a point $z \in \mathbf{C}$ if the inequality

$$\left| z - \frac{\alpha}{\beta} \right|^2 - \frac{1}{|\beta|^2} < \left| z - \frac{\lambda}{\mu} \right|^2 - \frac{1}{|\mu|^2}$$

is satisfied. This is, of course, an abuse of language because there may not be any points on $S_{\beta,\alpha}$ or $S_{\mu,\lambda}$ with coordinate z. However, if there is a point (z, ζ) on $S_{\mu,\lambda}$, the right side of the inequality is just $-\zeta^2$. Thus the left side is negative and so of the form $-\zeta'^2$. Clearly, $(z, \zeta') \in S_{\beta,\alpha}$ and $\zeta' > \zeta$. Therefore the definition agrees with the obvious meaning of covering whenever there is a point of $S_{\mu,\lambda}$ to be covered. The inequality is equivalent to

$$-2\mathscr{R}\left(\bar{z}\frac{\alpha}{\beta}\right) + \left|\frac{\alpha}{\beta}\right|^2 - \frac{1}{|\beta|^2} < -2\mathscr{R}\left(\bar{z}\frac{\lambda}{\mu}\right) + \left|\frac{\lambda}{\mu}\right|^2 - \frac{1}{|\mu|^2}.$$

Since this is (real) linear in z, we see that the set of z over which $S_{\beta,\alpha}$ covers $S_{\mu,\lambda}$ is an open half-plane $H(\alpha/\beta, \lambda/\mu)$. The boundary of this is a line $L(\alpha/\beta, \lambda/\mu)$. If $z \in L(\alpha/\beta, \lambda/\mu)$, we say that $S_{\beta,\alpha}$ and $S_{\mu,\lambda}$ agree over z (with the same abuse of language).

Define $G(\alpha_i + \gamma)$ to be the intersection of all $H(\alpha_i + \gamma, \alpha_j + \delta)$ for $j = 1,\ldots, n$, $\delta \in \mathcal{O}$, $\alpha_j + \delta \neq \alpha_i + \gamma$. This depends, of course, on the set $\{\alpha_1, \ldots, \alpha_n\}$. Thus $G(\alpha_i)$ is the set of all $z \in \mathbf{C}$ over which $S(\alpha_i)$ covers all other $S(\alpha_j + \gamma)$. Clearly, $G(\alpha_i + \gamma) = G(\alpha_i) + \gamma$. Similarly, let $F(\alpha_i)$ be the intersection of all $\bar{H}(\alpha_i, \alpha_j + \gamma)$ for $j = 1,\ldots, n$, $\gamma \in \mathcal{O}$, $\alpha_j + \gamma \neq \alpha_i$. This is the set over which $S(\alpha_i)$ covers or agrees with all other $S(\alpha_j + \gamma)$. Clearly, $G(\alpha_i)$ is an open convex set and $F(\alpha_i)$ is a closed convex set. It is also easy to see that $F(\alpha_i)$ and $G(\alpha_i)$ are bounded. In fact, $\bar{H}(\alpha_i, \alpha_i + 1) \cap \bar{H}(\alpha_i, \alpha_i - 1)$ is a vertical strip of finite width, and $\bar{H}(\alpha_i, \alpha_i + \omega) \cap \bar{H}(\alpha_i, \alpha_i - \omega)$ is a nonvertical strip of finite width. The intersection of these two strips is therefore bounded. It follows from this that $G(\alpha_i)$ and $F(\alpha_i)$ are intersections of only a finite number of $H(\alpha_i, \alpha_j + \gamma)$ and $\bar{H}(\alpha_i, \alpha_j + \gamma)$. In fact, if $S(\alpha_i)$, $S(\alpha_j + \gamma)$ have radii r and R and the distance from α_i to $\alpha_j + \gamma$ is d, then $L(\alpha_i, \alpha_j + \gamma)$ is perpendicular to the line from α_i to $\alpha_j + \gamma$ and cuts it at a distance $d/2 + (r^2 - R^2)/2d$ from α_i. Since $r, R \leqslant 1$, it is clear that for large d this line will not meet the bounded intersection of strips which we looked at above. In fact, this intersection is a parallelogram with corners $\alpha_i \pm \frac{1}{2} \pm \omega/2$, so we need only consider $\alpha_j + \gamma$ with $d \leqslant 1 + A$ where $A = \max(|1 + \omega|, |1 - \omega|)$ because for $d > 1 + \frac{1}{2}A$ we have $d/2 + (r^2 - R^2)/2d > d/2 - \frac{1}{2} > \frac{1}{2}A$ which is the distance from α_i to the farthest corner of our parallelogram.

In simple cases we can find the $G(\alpha_i)$, $F(\alpha_i)$ by drawing diagrams. I will now give an algorithm for handling the general case. We are given a finite collection of open half planes H_i, $i = 1,\ldots, m$ defined by inequalities $f_i(x) > a_i$ where f_i is a linear form in $x = (x_1, x_2)$. The closed half plane $\bar{H}_i$ is defined by $f_i(x) \geqslant a_i$. Let $G = \bigcap H_i$ and $F = \bigcap \bar{H}_i$. We must determine the structure of the convex sets F and G. Let L_i be the line $\bar{H}_i - H_i$ defined by $f_i(x) = a_i$. We can easily determine the intersection $J_i = L_i \cap F$. We need only choose a parametrization $x_\nu(t) = a_\nu + b_\nu t$, $\nu = 1, 2$ for this line. Substituting $x = x(t)$ in each inequality $f_j(x) \geqslant a_j$, $j \neq i$ we get a set of inequalities $t \geqslant c_j$ or $t \leqslant c_j$. The set of these defines J_i. Each J_i is either empty, a point, or a line segment, possibly of infinite length. Once the J_i are known, we can immediately determine the sets F and C. I will state the result in a form valid in a Euclidean space of any dimension. In the general case, of course, the algorithm is not as efficient because we must determine the J_i inductively by applying the algorithm in one dimension lower. The case where the L_i are all parallel is trivial and we will exclude it to simplify the statement.

PROPOSITION 8.1. *Let $H_1, \ldots, H_m$ be a finite collection of distinct open half spaces in an n-dimensional Euclidean space E. Let $L_i = \bar{H}_i - H_i$ be the hyperplane bounding H_i. We assume not all the L_i are parallel. Suppose $L_1, \ldots, L_r$ with $r \leqslant m$ are exactly the distinct hyperplanes so obtained, every L_i with $i > r$ being equal to one of these. Let $F = \bigcap \bar{H}_i$. Let $J_i = F \cap L_i$ and let $s = \max \dim J_i$. Then*

(1) *If $s \leqslant n - 2$, we have $F = J_i$ for every J_i of dimension s.*

(2) *If $s = n - 1$ and exactly one J_{i_0} has dimension $n - 1$, then $F = J_{i_0}$.*

(3) *If $s = n - 1$ and at least two J_i have dimension $n - 1$, then* $\dim F = n$ *and the boundary of F is the union of those J_i having dimension $n - 1$.*

Furthermore, $G = \bigcap H_i$ is empty in case (1) *and* (2). *In case* (3), $G = \operatorname{int} F$ *and $F = \bar{G}$. In all cases, F is bounded if and only if the J_i of dimension s are bounded.*

Proof. Let $J = \bigcup J_i$ and let $E' = E - J$, $F' = F - J$. Then F' is closed in E'. Also $F' \cap L_i = \emptyset$ so $F' \subset H_i$ for all i. Therefore $F' \subset G - J$. Since $F \cap G$ we have $F' - J = G - J$ which is open in E'. Therefore, F' is either empty or a union of components of E'. In case (1), $\dim J \leqslant n - 2$ so E' is connected [3]. If $F' = E'$, then $F = E$ since E' is dense in E and F is closed. This is clearly impossible. Thus $F = J$ in case (1). In case (2), let J' be the union of all J_i except J_{i_0}. Then

$\dim J' \leqslant n-2$. Now $E - L_{i_0}$ has two components, say E' and E'', each being an open half space. Since $\dim J' \leqslant n-2$, $E' - J'$ and $E'' - J''$ are connected and are the components of $E - L_{i_0} - J'$. Since $F - L_{i_0} - J' = F' - L_{i_0}$ is open and closed, it is either empty, $E' - J'$, $E'' - J'$ or $E - L_{i_0} - J'$. If $E' - J' \subset F' - L_{i_0}$, then $\bar{E}' \subset F$ since $E' - J'$ is dense in $\bar{E}'$. But $\bar{E}'$ is a half space. It is clearly impossible for F to contain a half space because by hypothesis, some L_j is not parallel to L_{i_0} and so $\bar{H}_j \cap \bar{E}'$ will be a proper subset of $\bar{E}'$. Therefore we must have $F - L_{i_0} - J' = \emptyset$ so $F \subset L_{i_0} \cup J'$ or $F = J$ since $F \cap L_{i_0} = J_{i_0}$.

It is now easy to see that if $F = J$, then $F = J_i$ for all J_i of dimension s. Since $F \cap L_i = J_i$, it will suffice to show that $F \subset L_i$. If not, let $p \in F$, $p \notin L_i$. The convex hull of $J_i \cup \{p\}$ lies in F. Since it is a non-degenerate cone with vertex p and base J_i, it has dimension $s+1$ so $\dim F \geqslant s+1$. This is impossible since $F = \bigcup J_i$ is a finite union of sets of dimension $\leqslant s$.

Now suppose two of the J_i say J_{i_0}, J_{i_1} have dimension $n-1$. Since $L_{i_0} \cap L_{i_1}$ has dimension $n-2$ (or is empty), $J_{i_1} \not\subset L_{i_0}$. Therefore F has a point $p \notin L_{i_0}$. The argument with the cone just given shows that $\dim F \geqslant n$. Since $n = \dim E$ and F is convex, standard theorems [3] show that $\operatorname{int} F \neq \emptyset$ and $\operatorname{int} F$ is dense in F. Now G is open and contained in F so $G \subset \operatorname{int} F$. If $p \in F - G$ then p lies on some L_i. Any neighborhood of p must meet the complement of $\bar{H}_i$, so $p \notin \operatorname{int} F$. This shows $\operatorname{int} F = G$ and $\bar{G} = F$. Since $J = F - G$, it is clear that $J = \partial F$. We must show that J is the union of the J_i with $\dim J_i = n-1$. Let J' be the union of all J_i with $\dim J_i \leqslant n-2$. Let $F' = \bigcap \bar{H}_i$, $G' = \bigcap H_i$ over those i with $\dim J_i = n-1$. Clearly $G \subset G'$, so $G' \neq \emptyset$ and $F' = \bar{G}'$. Now $F - J' - G \subset \bigcup L_i$ over those i with $\dim J_i = n-1$ since $F - J' - G \subset \bigcup J_i$ over these i. Now $G' \cap \bigcup L_i = \emptyset$ so

$$G' \cap (F - J' - G) = \emptyset \qquad \text{or} \qquad G' \cap F \subset J' \cup G.$$

But $G \subset G' \cap F$. Removing J', we get $G \subset G' \cap F - J' \subset G$ or $G = (G' - J') \cap F$ so G is closed in $G' - J'$. Now G' is homeomorphic to E [3] and $\dim J' \leqslant n-2$ so $G' - J'$ is connected. Since G is open and closed in $G' - J'$, and $G \neq \emptyset$ we must have $G = G' - J'$. But $G' - J'$ is dense in G' so $F = \bar{G} = \bar{G}' = F'$. Therefore $G = \operatorname{int} F = \operatorname{int} F' = G'$ so $J' \cap G' = \emptyset$. But $F' - G' = \bigcup F' \cap \bar{H}_i$ over those i with $\dim J_i = n-1$, i.e., $F - G = \bigcup J_i$ over those i with $\dim J_i = n-1$.

Finally, if F is bounded, so are all J_i. Conversely if all J_i of dimension

s are bounded, then in cases (1) and (2) we see that F is bounded. In case (3), $\partial F = J$ is bounded. To see that F is bounded we only have to observe that F is the convex hull of ∂F. In fact this is true of any closed convex set which is not the whole space or a half space. To see this, take two nonparallel support planes $f(x) = a$, $g(x) = b$ so that all $x \in F$ satisfy $f(x) \geqslant a$, $g(x) \geqslant b$. Choose v with $f(v) > 0$, $g(v) < 0$. If $x \in F$, the line $x + tv$, $t \in R$, will meet F in a finite line segment since $f(x + tv) < a$ for $t \to -\infty$ and $g(x + tv) < a$ for $t \to +\infty$. The ends of e clearly lie on ∂F.

We now return to the case of the plane and our sets $F(\alpha_i)$, $G(\alpha_i)$. By Proposition 8.1, if $G(\alpha_i) \neq \emptyset$, then $F(\alpha_i) - G(\alpha_i)$ is a polygon.

PROPOSITION 8.2. *The nonempty open 2-cells $G(\alpha_i)$ and their translates $G(\alpha_i + \gamma)$ for $\gamma \in \mathcal{O}$ define a convex polyhedral cell subdivision of* $\mathbf{C}$, *the edges and vertices being the natural edges and vertices of the polygons $F(\alpha_i + \gamma) - G(\alpha_i + \gamma)$.*

Proof. It is clear from the definition that the 2-cells $G(\alpha_i + \gamma)$ are all disjoint. Let e be an edge of $F(\alpha_i)$. We must show that e is also an edge of some $F(\alpha_j + \gamma)$ lying on the opposite side of e from $G(\alpha_i)$. By Proposition 8.1, e lies on one of the lines $L(\alpha_i, \alpha_j + \gamma)$. As we observed earlier only a finite number of such lines will meet any given bounded set. In particular, only a finite number contain e. If e lies on $L(\alpha_i, \alpha_j + \gamma)$ and on $L(\alpha_i, \alpha_k + \delta)$, then $S(\alpha_i)$ agrees with $S(\alpha_j + \gamma)$ over e and also with $S(\alpha_k + \delta)$. Thus $S(\alpha_j + \gamma)$ and $S(\alpha_k + \delta)$ agree on e, so on the side of e away from $G(\alpha_i)$, one of $S(\alpha_j + \gamma)$ and $S(\alpha_k + \delta)$ covers the other. Suppose $S(\alpha_j + \gamma)$ is the one which covers all the others. We will show that e lies on the edge of $F(\alpha_j + \gamma)$. Let U be a small neighborhood of the open edge e and let $V = U - F(\alpha_i)$. We must show that if U is small enough, then $S(\alpha_j + \gamma)$ covers all other $S(\alpha_k + \delta)$ over V. There are only a finite number of $\alpha_k + \delta$ for which this is not clear. Suppose first that the open edge e does not meet the line $L = L(\alpha_j + \gamma, \alpha_k + \delta)$. Take U small enough so $U \cap L = \emptyset$ and U lies on one side of L. If $S(\alpha_j + \gamma)$ did not cover $S(\alpha_k + \delta)$ over V, then $S(\alpha_k + \delta)$ would cover $S(\alpha_j + \gamma)$ over V. Since it cannot cover $S(\alpha_i)$ over $G(\alpha_i)$, we see that e must lie on $L(\alpha_i, \alpha_k + \delta)$ but this is impossible by the choice of $S(\alpha_j + \gamma)$.

Next, we must show that it is not possible for any $L = L(\alpha_i + \gamma, \alpha_k + \delta)$ to meet e. But, if this were so, then $S(\alpha_k + \delta)$ would cover $S(\alpha_j + \gamma)$ over some point of e. Since $S(\alpha_j + \gamma)$ and $S(\alpha_i)$ coincide over e, $S(\alpha_k + \delta)$

would cover $S(\alpha_i)$ over some point $p \in e$ and hence over a neighborhood W of p. This is impossible since W will meet $G(\alpha_i)$.

We have now shown that all points sufficiently near e on the opposite side from $G(\alpha_i)$ must lie in $F(\alpha_j + \gamma)$. Therefore e lies on the boundary of $F(\alpha_j + \gamma)$. Since e is a line segment, it lies on some edge e' of $F(\alpha_j + \gamma)$. Reversing the argument, e' lies on an edge e'' of some other 2-cell which can only be $F(\alpha_i)$. Since $e \subset e' \subset e''$ and e, e'' are edges of $F(\alpha_i)$, we have $e = e''$ so $e = e'$.

We can deduce from this a simple and useful check on the accuracy of our calculations.

COROLLARY 8.3. *Let $\alpha_1, \ldots, \alpha_n \in K$ and let S be the set of all $\alpha_i + \gamma$, $1 \leqslant i \leqslant n$, $\gamma \in \mathcal{O}$. Let P be a convex polyhedral cell division of* $\mathbf{C}$. *Suppose we are given a one-to-one correspondence between the elements of S and 2-cells of P, the element $\alpha \in S$ corresponding to the 2-cell $e(\alpha)$. Suppose that whenever $e(\alpha)$ and $e(\beta)$ are adjacent 2-cells, their common boundary lies on the line $L(\alpha, \beta)$. Then P is the subdivision given by Proposition* 8.2.

Proof. Clearly $e(\alpha) = \bigcap H(\alpha, \beta)$ with β running over all elements of S such that $e(\alpha)$ and $e(\beta)$ are adjacent. But $F(\alpha) = \bigcap H(\alpha, \beta)$ with β running over all elements of $S - \{\alpha\}$ so $F(\alpha) \subset e(\alpha)$. Therefore, no interior point of $e(\alpha)$ can lie in any $F(\beta)$ for $\beta \neq \alpha$. Since $C = \bigcup F(\alpha)$, we must have $\operatorname{int} e(\alpha) \subset F(\alpha)$. Therefore $e(\alpha) = F(\alpha)$.

We can now determine when $B(\alpha_1, \ldots, \alpha_n) = B$. Using Proposition 8.1 we can determine the $F(\alpha_i)$ and thus obtain the cell structure of Proposition 8.2. Now projection gives a homeomorphism $\partial B(\alpha_1, \ldots, \alpha_n) \to \mathbf{C}$. We use this to transfer the cell structure on $\mathbf{C}$ to $\partial B(\alpha_1, \ldots, \alpha_n)$.

PROPOSITION 8.4. *We have $B(\alpha_1, \ldots, \alpha_n) = B$ if and only if no vertex of $\partial B(\alpha_1, \ldots, \alpha_n)$ can be covered by any $S_{\mu,\lambda}$ with $(\mu, \lambda) = \mathcal{O}$.*

In other words, no vertex v of $\partial B(\alpha_1, \ldots, \alpha_n)$ can lie strictly below any $S_{\mu,\lambda}$ with $(\mu, \lambda) = \mathcal{O}$.

Proof. If $B(\alpha_1, \ldots, \alpha_n) = B$, the condition is satisfied because *no* point of B can be covered by any $S_{\mu,\lambda}$ with $(\mu, \lambda) = \mathcal{O}$. Suppose now that no vertex can be covered. Suppose some point $p \in \partial B(\alpha_1, \ldots, \alpha_n)$ can be covered. This point lies over some $F(\alpha_i)$ and so lies on $S(\alpha_i)$. Let $p = (z, \zeta)$ so $z \in F(\alpha_i)$. If $S(\gamma)$ covers p, then $z \in H(\gamma, \alpha_i)$. Now the complement of $H(\gamma, \alpha_i)$ is the convex set $\bar{H}(\alpha_i, \gamma)$. If all vertices of $F(\alpha_i)$ lie in $\bar{H}(\alpha_i, \gamma)$, so does $F(\alpha_i)$ since $F(\alpha_i)$ is the convex hull of its vertices.

Thus some vertex of $F(\alpha_i)$ lies in $H(\gamma, \alpha_i)$. The corresponding vertex of $\partial B(\alpha_1, ..., \alpha_n)$ lies on $S(\alpha_i)$ and so is covered by $S(\gamma)$.

The condition of Proposition 8.4 is very easy to check. It is clear that the condition need only be checked at a set of representatives of the vertices modulo translations by $\mathcal{O}$. Thus we have only a finite number of points to check. We first determine all singular points as in Section 7. If any vertex of $\partial B(\alpha_1, ..., \alpha_n)$ lies in the plane $\zeta = 0$ and is not singular, then it can be covered. We can find some $\zeta(\gamma)$ covering it by the methods of Section 7 or by trial and let $\alpha_{n+1} = \gamma$. Suppose that all vertices with $\zeta = 0$ are singular. Since the singular points cannot be covered, we have only to check the remaining vertices. Let $v = (z, \zeta)$, $\zeta > 0$ be such a vertex. We have to find all μ, λ with $| \mu z - \lambda |^2 + \zeta^2 | \mu |^2 < 1$. Since $| \mu | < \zeta^{-1}$, there are only a finite number of possibilities for μ. For each μ, there are only a finite number of possibilities for λ. Therefore we can enumerate all possibilities and check whether any satisfy $(\mu, \lambda) = \mathcal{O}$.

If in the course of this work we also determine all μ, λ such that $| \mu z - \lambda |^2 + \zeta^2 | \mu |^2 = 1$, we can then determine, in most cases, which $S_{\mu,\lambda}$ contain a given edge e. This of course is only of interest if we do find that $\partial B(\alpha_1, ..., \alpha_n) = B$.

LEMMA 8.5. *If e is a geodesic segment of H and S is a hemisphere in H with center in the plane $\zeta = 0$, then $e \subset S$ if and only if both endpoints of e lie on S.*

This continues to hold if the endpoints lie in the plane $\zeta = 0$ if lying on S is interpreted as lying on the closure of S.

Proof. Let C be the vertical semicircle on which e lies. Then C lies in a vertical plane P and $P \cap S = C'$ is also a vertical semicircle. Both C and C' have centers in the plane $\zeta = 0$. If C and C' do not coincide, they can meet in at most two points. Suppose $C \cap C' = \{p, q\}$, $p \neq q$. The line pq lies in P and is perpendicular to the line joining the centers of C and C'. Therefore pq is a vertical line and so can meet C in at most one point. Thus if C and C' do not coincide, they have at most one point in common so it is not possible for both ends of e to lie on S.

We can now use this lemma to determine which $S_{\mu,\lambda}$ contain a given edge e. If both ends of e are nonsingular we only have to see which $S_{\mu,\lambda}$ contain both ends. If one end s of e is singular, we look for those $S_{\mu,\lambda}$ containing the other end and check which of them also contain s. If both ends of e are singular, we must fall back on Lemma 4.19. Alternatively, we can note that if s, s' are singular and lie on $S_{\mu,\lambda}$ then $|\mu| \leqslant 2|s - s'|^{-1}$. This gives us only a finite number of possibilities to check.

In all the above calculations, we can, of course, make use of the symmetries considered in Section 4 to reduce the number of calculations necessary provided we are careful to choose the set $\{\alpha_1, ..., \alpha_n\}$ invariant under π (modulo $\mathcal{O}$).

We will now show that it is possible to calculate rather easily an upper bound for the values of μ for which $S_{\mu,\lambda}$ can meet B. This will make it possible to estimate in advance how many α_i will have to be tried before we arrive at $B(\alpha_1, ..., \alpha_n) = B$. Since the bound obtained will be much too large in general, I will not give a value for it but will merely show how it can be obtained. The proof of Theorem 3.13 will not give us such a bound directly since this proof relied on a compactness argument.

The first step will be to estimate the values of the constants occurring in the proof of Theorem 2.1. Given any $z \in \mathbf{C}$, $z \notin K$ and any positive integer M, the proof of Lemma 2.2 shows that we can find $\alpha, \beta \in \mathcal{O}$ with $|\beta z - \alpha| \leqslant DM^{-1}$ and $|\beta| \leqslant DM$, where $D = \frac{1}{2}\sqrt{|\Delta| + 9}$ with Δ the discriminant of K. We now let the ideals $\mathfrak{c}_i$ be the ideals $(q, a + \omega)$ of Theorem 7.3 and choose $\gamma_i = q$ in $(q, a + \omega)$. The constant A_i in Lemma 2.3 is $\leqslant |q| D + |a + \omega|$ where D is as above. Since $q \leqslant \sqrt{m/3}$ and $|a| \leqslant \frac{1}{2}q$ we can easily find the maximum $A = \max A_i$. Also $G = \max |\gamma_i| \leqslant \sqrt{m/3}$ and $N_0 = \max N\mathfrak{c}_i \leqslant \sqrt{m/3}$.

We now proceed as follows: For each singular point s_i we follow the method of Theorem 3.13 and get a small neighborhood of radius ϵ_i over which $S_{\mu,\lambda}$ cannot meet B unless $|\mu| \leqslant m_i$. For each $\beta \in \mathcal{O}$ with $|\beta| \leqslant 4G$, and each nonsingular α/β, there are $\mu, \lambda \in \mathcal{O}$ with $(\mu, \lambda) = \mathcal{O}$ $|\mu(\alpha/\beta) - \lambda| = |\gamma/\beta| < 1$ where $\gamma = |\mu\alpha - \beta\lambda|$. Now $N(\gamma) < N(\beta)$, so $N(\gamma) \leqslant N(\beta) - 1$ or $|\gamma/\beta|^2 \leqslant 1 - |\beta|^{-2} \leqslant 1 - 1/16G^2$. Also we can take $|\mu| \leqslant A|\beta| \leqslant 4GA$. Therefore

$$\left|\frac{\alpha}{\beta} - \frac{\lambda}{\mu}\right|^2 \leqslant \frac{1}{|\mu|^2}\left|\frac{\gamma}{\beta}\right|^2 \leqslant \frac{1}{|\mu|^2} - \frac{1}{64G^3A}.$$

Clearly we can find $\delta, \eta > 0$ depending only on G and A such that for each such α/β, the points (z, ζ) on $S_{\mu,\lambda}$ with $|z - \alpha/\beta| < \delta$ have $\zeta \geqslant \eta$. Let $\epsilon = \min(\epsilon_i, \delta)$, $m = \max(m_i, \eta^{-1})$. Therefore if α/β is singular or has $|\beta| \leqslant 4G$, no $S_{\mu,\lambda}$ can meet B over the ϵ-neighborhood of α/β unless $|\mu| \leqslant m$. Let U be the union of all these ϵ-neighborhoods.

Now choose a positive integer M so large that $DA/M < 1/4$ and $D/M < \epsilon$. If $z \in \mathbf{C}$, $z \notin K$, $z \notin U$, we can find $\beta, \alpha \in \mathcal{O}$, so $|\beta| \leqslant DM$,

$|\beta z - \alpha| < DM^{-1} < \epsilon$. Since $\beta \in \mathcal{O}$, we have $|z - \alpha\beta^{-1}| < \epsilon |\beta|^{-1} < \epsilon$. Since $z \notin U$, it follows that $|\beta| > 4G$. Now find $\mu, \lambda \in \mathcal{O}$ with $(\mu, \lambda) = \mathcal{O}$, $|\mu\alpha - \lambda\beta| = |\gamma| \leqslant G$ and $|\mu| \leqslant A|\beta| \leqslant ADM$. Then

$$\left| z - \frac{\lambda}{\mu} \right| \leqslant \left| z - \frac{\alpha}{\beta} \right| + \left| \frac{\alpha}{\beta} - \frac{\lambda}{\mu} \right|$$

$$\leqslant \frac{D}{M|\beta|} + \frac{G}{|\beta||\mu|} \leqslant \frac{DA}{M|\mu|} + \frac{G}{4G|\mu|} < \frac{1}{2\mu}.$$

Therefore the point $(z, \zeta) \in S_{\mu,\lambda}$ has

$$\zeta^2 = \frac{1}{|\mu|^2} - \left| z - \frac{\lambda}{\mu} \right|^2 \geqslant \frac{1}{|\mu|^2} - \frac{1}{4|\mu|^2} \geqslant \frac{3}{4|\mu|^2},$$

so $\zeta > \sqrt{3}/2|\mu| \geqslant \sqrt{3}/2ADM$. But these points are dense in the part of ∂B lying over the complement of U because the vertical projection $\partial B \to \mathbf{C}$ is a homeomorphism by Theorem 3.13. Therefore all points $(z, \zeta) \in B$ with $z \notin U$ satisfy $\zeta \geqslant \sqrt{3}/2ADM$. Thus if $S_{\mu,\lambda}$ meets B, either it meets B over U and $|\mu| \leqslant m$ or it meets B over the complement of U and $|\mu| \leqslant 2ADM/\sqrt{3}$.

9. Further Remarks on Determining B

In determining B for small values of m, the simplest procedure is to draw a diagram of the cell decomposition of $\mathbf{C}$. In order to record the information so obtained, it is sufficient to draw that part of the cell complex lying in a fundamental domain F for $\mathbf{C}$ under the translations by elements of $\mathcal{O}$. Writing $z = \xi + i\eta$, we can choose F to be given by $-\frac{1}{2} \leqslant \xi \leqslant \frac{1}{2}$ and $-\sqrt{m}/2 \leqslant \eta \leqslant \sqrt{m}/2$ if $m \not\equiv 3 \bmod 4$ or $-\sqrt{m}/4 \leqslant \eta \leqslant \sqrt{m}/4$ if $m \equiv 3 \bmod 4$. Making use of the symmetries considered in Section 4 we can even use a smaller region. Let π be the four group generated by $e(z) = -z$ and $c(z) = \bar{z}$. In terms of ξ and η we have $e(\xi, \eta) = (-\xi, -\eta)$, $c(\xi, \eta) = (\xi, -\eta)$, $ec(\xi, \eta) = (-\xi, \eta)$. Therefore to record all the information needed to give B, it will suffice to draw that portion of the cell decomposition lying in the region Q defined by $0 \leqslant \xi \leqslant \frac{1}{2}$ and $0 \leqslant \eta \leqslant \sqrt{m}/2$ for $m \not\equiv 3 \bmod 4$ or $0 \leqslant \eta \leqslant \sqrt{m}/4$ for $m \equiv 3 \bmod 4$. Of course, to actually see what B looks like it is better to make a diagram of a much larger part of the plane. However, for simplicity, we will just give the diagram of Q for each value of m considered in the following sections. The full diagram is then easily found by filling up the plane with blocks obtained from Q by the transformations of π and $\mathcal{O}$.

Since Q itself has sides, the diagram of Q will contain some spurious edges and vertices formed by those portions of the sides of Q which do not lie on edges or vertices of the cell decomposition of **C**. In drawing the diagrams, we will mark those edges and vertices which are true edges (or parts of true edges) and vertices. However, it is very easy to recognize which edges and vertices of Q are true ones even if these are not marked. For those lying on the vertical sides of Q this can be done using the fact that the reflections in these vertical lines are symmetries of B, the reflection about $\xi = 0$ sending z to $ec(z)$ and that in $\xi = \frac{1}{2}$ sending z to $1 + ec(z)$. Suppose e is an edge of Q lying on a vertical side. Then e is an edge of some 2-cell $Q \cap F(\alpha)$. If r is the reflection in the side on which e lies, the other cell which e bounds will be $r(Q \cap F(\alpha)) = rQ \cap F(r(\alpha))$. If $r(\alpha_i) = \alpha_i$, this is part of the same 2-cell $F(\alpha)$ and e is not a true edge. If, however, $r(\alpha) \neq \alpha$, then e is part of $F(\alpha) \cap F(r(\alpha))$ and so is a true edge or part of one. In other words, if α does not lie on the line of which e is a segment, then e is a true edge or part of one. If, however, α lies on the line of e, then e is a spurious edge.

Suppose now that v is a vertex lying on a vertical side of Q but not a corner of Q. If two true edges in Q meet at v, then v is certainly a true vertex. If only one true edge e passes through v, then e does not lie on the vertical side of Q otherwise v would not be a vertex. Note that if v is the end of some edge e' outside Q, then re' lies in Q and passes through v. Thus one cannot miss finding a vertex by only looking at Q. Now, the only edge which can meet v from outside Q is $r(e)$. If e and $r(e)$ are parallel, then v is clearly not a true vertex, whereas if e and $r(e)$ are not parallel then v must be a true vertex. This last case however cannot occur. Let $Q \cap F(\alpha)$ be the 2-cell of Q immediately above e and $Q \cap F(\beta)$ the one immediately below e. Since there is no true edge on the vertical side meeting v, we have $r(\alpha) = \alpha$, $r(\beta) = \beta$. Thus $e \cup r(e)$ lies on the intersection $F(\alpha) \cap F(\beta)$ but this is a straight line segment.

Thus we have shown that a vertex v lying on a vertical side of Q and not a corner of Q will be a true vertex if and only if it lies on at least two true edges in Q. Furthermore, if v is a spurious vertex, it lies on a single true edge e of Q, e is horizontal, and e is half of a true edge, the full edge being $e \cup r(e)$ where r is the reflection in the vertical side of Q containing v. Note that in the latter case, the other end of e in Q must be a true vertex otherwise the union of all $e + n$, $ec(e) + n$ for $n \in Z$ would be composed entirely of edges but would have no vertex on it.

We can now do a similar analysis of the top and bottom of Q. This is a bit more complicated when $m \equiv 3 \bmod 4$. In this case, the reflection in the top is not a symmetry and we must use a glide reflection instead. However, we can avoid all this by actually determining all edges and vertices which can meet the top or bottom of Q. The few exceptions which will occur in the general results will be treated in detail later. We will use the following Lemma which is another variant of the idea expressed by Lemma 4.4 and Lemma 8.5.

LEMMA 9.1. *Let S be a hemisphere in H with center in the plane $\zeta = 0$. Let e be an open geodesic segment in H. If some point of e is covered by or lies on S, then either $e \subset S$ or at least one end of e is covered by S.*

As always we say a point (z, ζ) is covered by S if there is a point (z', ζ') on S with $z' = z$, $\zeta' > \zeta$.

Proof. As in the proof of Lemma 4.4, e lies on a vertical semicircle C, P is the vertical plane through C, $C' = P \cap S$, and we know that either $C = C'$ or $C \cap C'$ has at most one point. If $e \not\subset S$, then $C \neq C'$. If $C \cap C' = \{p\}$, then on one side of p, C covers C' while on the other side C' covers C. Since e contains a point of this latter side of C, one end (at least) of e is also on this side. The case $C \cap C' = \emptyset$ is even easier.

LEMMA 9.2. *Over the line segment $\eta = 0$, $0 \leqslant \xi \leqslant \frac{1}{2}$, $S_{1,0}$ covers all other $S_{\mu,\lambda}$ with $(\mu, \lambda) = \mathcal{O}$ except that $S_{1,0}$ and $S_{1,1}$ agree over the point $\eta = 0$, $\xi = \frac{1}{2}$.*

Proof. This is almost obvious but I will use the general methods developed above to illustrate their use. We first determine which $S_{\mu,\lambda}$ cover or meet the ends of the geodesic segment e which lies on $S_{1,0}$ over the line segment $\eta = 0$, $0 \leqslant \xi \leqslant \frac{1}{2}$. The ends of e have $(z, \zeta) = (0, 1)$ and $(z, \zeta) = (\frac{1}{2}, \sqrt{3}/2)$. If $(0, 1)$ lies on or under $S_{\mu,\lambda}$ we must have $| \mu \cdot 0 - \lambda |^2 + | \mu |^2 \leqslant 1$. Therefore $\mu = \pm 1$, $\lambda = 0$. If $(\frac{1}{2}, \sqrt{3}/2)$ lies on or under $S_{\mu,\lambda}$, we have $| \mu \cdot \frac{1}{2} - \lambda |^2 + \frac{3}{4} | \mu |^2 \leqslant 1$. Again $\mu = 1$ and $| \frac{1}{2} - \lambda |^2 \leqslant \frac{1}{4}$, so $\lambda = 0$ or 1. In each case, the inequality $| \mu z - \lambda |^2 + \zeta^2 | \mu |^2 \leqslant 1$ is actually an equality. Thus neither end of e is covered by any $S_{\mu,\lambda}$. By Lemma 8.5, no point of e other than the ends can lie on or be covered by any $S_{\mu,\lambda}$ which does not contain e. If it contains e, it must contain the ends of e and so can only be $S_{1,0}$. The remaining statements are clear from what we have done.

COROLLARY 9.3. *There are no vertices or edges on the line* $\eta = 0$. *The only edges which meet this line are those giving the boundary between the* $S_{1,n}$, $S_{1,n+1}$ *for* $n \in \mathbf{Z}$. *These edges are all equivalent modulo* $\mathcal{O}$. *They lead to the edge relation* $(TA)^3 = J$ *in* $SL(2, \mathcal{O})$ *where*

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The inverse relation for A *is* $A^2 = J$, A *being the generator corresponding to the* 2*-cell* $F(0/1)$.

Proof. It is clear from Lemma 9.2 that there are no edges on $\eta = 0$. The $S_{1,n}$ cover all other $S_{\mu,\lambda}$ over a neighborhood of $\eta = 0$ since covering always occurs over an open set. Thus the only edges which can meet $\eta = 0$ are those defined by the $S_{1,n}$, i.e., those of the type indicated. If e is the edge between $S_{1,0}$ and $S_{1,1}$, the only λ/μ with $e \subset S(\lambda/\mu)$ are $0/1$ and $1/1$. As the corresponding matrices, we choose A and $AT = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. This yields the relation $[A^{-1}][A(AT)^{-1}][AT] = 1$. Now

$$A(AT)^{-1} = AT^{-1}A^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = TAT,$$

so our relation is $A^{-1}TATAT = 1$ or $(AT)^3 = A^2$, i.e., $(AT)^3 = J$.

We now turn over attention to the top of Q. The result depends on the class of m modulo 8.

LEMMA 9.4. *Suppose* $m \equiv 1 \bmod 4$ *and* $m \geqslant 5$. *Let* $\omega = \sqrt{-m}$. *Then* $(2, \omega) = \mathcal{O}$ *and* $(\omega + 1)/2$ *is singular. Over the top of* Q, *i.e.,* $0 \leqslant \xi \leqslant \frac{1}{2}$, $\eta = \sqrt{m}/2$. $S_{2,\omega}$ *covers all other* $S_{\mu,\lambda}$ *except at the singular point* $(\omega + 1)/2 = (\frac{1}{2}, \sqrt{m}/2)$.

Proof. We have $(2, \omega) = \mathcal{O}$ since it contains $\omega^2 = -m$ which is odd. The singularity of $(\omega + 1)/2$ follows from Corollary 7.4 since $m \geqslant 5$. The geodesic segment e on $S_{2,\omega}$ over the top of Q has ends $(\omega/2, \frac{1}{2})$ and $((\omega + 1)/2, 0)$. No $S_{\mu,\lambda}$ can cover the singular point $(\omega + 1)/2$. If $(\omega/2, \frac{1}{2})$ is covered by or lies on $S_{\mu,\lambda}$, we have $|\mu(\omega/2) - \lambda|^2 + \frac{1}{4}|\mu|^2 \leqslant 1$ or $|\mu\omega - 2\lambda|^2 + |\mu|^2 \leqslant 4$. In particular, $|\mu|^2 \leqslant 4$. If $\mu = a + b\omega$, $|\mu|^2 = a^2 + mb^2$. Since $m \geqslant 5$, we must have $b = 0$ and $\mu = 1$ or 2 (up to sign). If $\mu = 1$, $|\omega - 2\lambda|^2 + 1 \leqslant 4$. This is impossible since $\omega - 2\lambda = x + y\omega$ with $y \neq 0$. If $\mu = 2$, $|2\omega - 2\lambda|^2 + 4 \leqslant 4$ so $\lambda = \omega$. The conclusion now follows as in Lemma 9.2.

COROLLARY 9.5. *If $m \not\equiv 3 \bmod 4$, m is odd, and $m \geqslant 5$, then ther are no edges on the line $\eta = \sqrt{m}/2$. The only vertices on this line are the singular points $(1 + \omega)/2 + n$, $n \in Z$. If an edge meets this line, it meets it at one of these singular points.*

LEMMA 9.6. *Suppose $m \equiv 2 \bmod 4$, and $m \geqslant 6$. Let $\omega = \sqrt{-m}$. Then $(2, \omega + 1) = \mathcal{O}$ and $\omega/2$ is singular. Over the top of Q, $0 \leqslant \xi \leqslant \frac{1}{2}$, $\eta = \sqrt{m/2}$, $S_{2,\omega+1}$ covers all other $S_{\mu,\lambda}$ except at the singular point $\omega/2 = (0, \sqrt{m/2})$.*

The proof is almost exactly the same as that of Lemma 9.4.

COROLLARY 9.7. *If $m \not\equiv 3 \bmod 4$, m is even, and $m \geqslant 6$, then there are no edges on the line $\eta = \sqrt{m}/2$. The only vertices on this line are the singular points $\omega/2 + n$, $n \in Z$. If an edge meets this line, it meets it at one of these singular points.*

We now examine the cases where $m \equiv 3 \bmod 4$, so $m \equiv -1$ or $3 \bmod 8$. As before, we let $\omega = (1 + \sqrt{-m})/2$.

LEMMA 9.8. *If $m \equiv 3 \bmod 8$ and $m \geqslant 19$, then over the top of Q, i.e., $\eta = \sqrt{m}/4$, $0 \leqslant \xi \leqslant \frac{1}{2}$, $S_{2,\omega}$ covers all other $S_{\mu,\lambda}$ with $(\mu, \lambda) = \mathcal{O}$ except that $S_{2,1+\omega}$ and $S_{2,\omega}$ agree over the point $\eta = \sqrt{m}/4$, $\xi = \frac{1}{2}$, and $S_{2,\omega-1}$ and $S_{2,\omega}$ agree over the point $\eta = \sqrt{m}/4$, $\xi = 0$.*

Proof. We have $N\omega = (1 + m)/4$, $N(\omega + 1) = (9 + m)/4$, $N(\omega - 1) = (1 + m)/4$. Since these are all odd, we see that $(2, \omega) = (2, \omega + 1) = (2, \omega - 1) = \mathcal{O}$. The top of Q has endpoints

$$(0, \sqrt{m}/4) = \frac{2\omega - 1}{4} \quad \text{and} \quad \left(\frac{1}{2}, \frac{\sqrt{m}}{4}\right) = \frac{2\omega + 1}{4}.$$

The geodesic segment e lying on $S_{2,\omega}$ over the top of Q has ends

$$(z, \zeta) = \left(\frac{2\omega - 1}{4}, \frac{\sqrt{3}}{4}\right), \left(\frac{2\omega - 1}{4}, \frac{\sqrt{3}}{4}\right).$$

To find which $S_{\mu,\lambda}$ cover or contain these endpoints, we have to find the μ, λ with $| \mu((2\omega - 1)/4) - \lambda |^2 + \frac{3}{16} | \mu |^2 \leqslant 1$, i.e.,

$$| \mu(2\omega - 1) - 4\lambda |^2 + 3 | \mu |^2 \leqslant 16 \quad \text{and} \quad | \mu(2\omega + 1) - 4\lambda |^2 + 3 | \mu |^2 \leqslant 16.$$

Now for $a, b \in \mathbf{Z}$, $|a + b\omega|^2 \geqslant b^2(m/4)$. Since $m \geqslant 19$, $|a + b\omega|^2 \geqslant 19$ for $|b| \geqslant 2$. For $b = 1$, we have $|a + \omega|^2 = (a + \frac{1}{2})^2 + m/4 = \frac{1}{4}[(2a + 1)^2 + m]$. Therefore $|\mu|^2 \leqslant 5\frac{1}{3}$ implies $\mu = 1, 2, \omega, \omega - 1$, the latter two values only occurring for $m = 19$. For $\mu = 1$, $|2\omega \pm 1 - 4\lambda|^2 + 3 \leqslant 16$ is impossible since $2\omega \pm 1 - 4\lambda = a + b\omega$ with $b \equiv 2 \bmod 4$ and so $|b| \geqslant 2$. For $\mu = 2$, $|4\omega + 2 - 4\lambda|^2 + 12 \leqslant 16$ can only hold for $\lambda = \omega$ or $\omega + 1$ and in each case we get equality. Similarly, $|4\omega - 2 - 4\lambda|^2 + 12 \leqslant 16$ can only hold for $\lambda = \omega$ or $\omega - 1$. We are now finished except for the case $m = 19$ where $\mu = \omega$ or $\omega - 1$ can occur. Since $|\mu|^2 = 5$, in this case we must have $|\mu(2\omega \pm 1) - 4\lambda|^2 \leqslant 1$. But if $\mu(2\omega \pm 1) - 4\lambda = a + b\omega$, we have $a + b\omega \equiv \pm\mu \bmod 2$, so $b \equiv \pm 1 \bmod 2$. Thus $|b| \geqslant 1$, so $|a + b\omega|^2 \geqslant \omega/4 > 4$.

COROLLARY 9.9. *For $m \equiv 3 \bmod 8$, $m \geqslant 19$, there are no vertices or edges on the line $\eta = \sqrt{m}/4$. The only edges which meet this line are those giving the boundary between the $S_{2,\omega+n}$, $S_{2,\omega+n+1}$ for $n \in Z$. These edges fall into two classes under translations by $\mathcal{O}$. They yield the two edge relations $B^3 = J$ and $(T^{-1}B)^3 = 1$, where*

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 - \omega & k \\ 2 & \omega \end{pmatrix}$$

with $k = \frac{1}{8}(m - 3)$.

Proof. The first part follows as in Corollary 9.9. We must now find the edge relations. For the edge between $S_{2,\omega}$ and $S_{2,\omega+1}$ we choose $\tau_1 = B$ and $\tau_2 = JB^{-1}T = \left(\begin{smallmatrix} -\omega & k-\omega \\ 2 & \omega+1 \end{smallmatrix}\right)$. The relation is $[B^{-1}][BT^{-1}BJ][JB^{-1}T] = 1$. But $BT^{-1}BJ = \left(\begin{smallmatrix} 2-\omega & * \\ 2 & \omega+1 \end{smallmatrix}\right) = JTB^{-1}T$, so our relation is $(B^{-1}T)^3 = 1$. For the edge between $S_{2,\omega}$ and $S_{2,\omega-1}$, we choose $\tau_1 = B$ and $\tau_2 = JB^{-1} = \left(\begin{smallmatrix} -\omega & k \\ 2 & \omega-1 \end{smallmatrix}\right)$. The relation is $[B^{-1}][BBJ][JB^{-1}] = 1$ But $B^2 = \left(\begin{smallmatrix} -\omega & * \\ 2 & \omega-1 \end{smallmatrix}\right) = JB^{-1}$ so our relation is $(B^{-1})^3 = J$.

LEMMA 9.10. *For $m \equiv -1 \bmod 8$, $m \geqslant 23$, the points $\omega/2$ and $(\omega + 1)/2$ are singular. Over the top of Q, $S_{4,2\omega-1}$ and $S_{4,2\omega+1}$ cover all other $S_{\mu,\lambda}$ except at the singular point $\omega/2$.*

Proof. That $\omega/2$ and $(\omega + 1)/2$ are singular follows from Corollary 7.4. Since $N(2\omega - 1) = m$ and $N(2\omega + 1) = m + 4$ are odd, we see that $(4, 2\omega - 1) = \mathcal{O} = (4, 2\omega + 1)$. The geodesic segment on $S_{4,2\omega-1}$

lying over the top of Q has endpoints $(z, \zeta) = ((2\omega - 1)/4, \frac{1}{4}), (\omega/2, 0)$ while that over $S_{4,2\omega+1}$ has endpoints $(z, \zeta) = ((2\omega + 1)/4, \frac{1}{4}), (\omega/2, 0)$. We have to look for $\mu, \lambda \in \mathcal{O}$ satisfying $| \mu((2\omega \pm 1)/4) - \lambda |^2 + \frac{1}{16} | \mu |^2 \leqslant 1$ or $| \mu(2\omega \pm 1) - 4\lambda |^2 + | \mu |^2 \leqslant 16$. We must show that there are no solutions with $(\mu, \lambda) = \mathcal{O}$ other than $\mu = 4$, $\lambda = 2\omega \pm 1$. It will suffice to consider one of these inequalities since if

$$| \mu(2\omega - 1) - 4\lambda |^2 + | \mu |^2 \leqslant 16, \quad \text{then} \quad | \mu(2\omega + 1) - 4\lambda' |^2 + | \mu |^2 \leqslant 16$$

where $\lambda' = \mu\omega - \lambda$. In fact, $\mu(2\omega + 1) - 4\lambda' = -[\mu(2\omega - 1) - 4\lambda]$. Note also that $(\mu, \lambda') = (\mu, \lambda)$.

Now consider $| \mu(2\omega - 1) - 4\lambda |^2 + | \mu |^2 \leqslant 16$. If $a, b \in \mathbf{Z}$, we have $| a + b\omega |^2 \geqslant b^2(m/4)$. If $| b | \geqslant 2$, $| a + b\omega | \geqslant 23$ since $m \geqslant 23$. Thus $\mu = a$ or $a + \omega$ (up to sign). Also $| a + \omega |^2 = \frac{1}{4}[(2a + 1)^2 + m]$, so if $| a + \omega |^2 \leqslant 16$, we have $(2a + 1)^2 + m \leqslant 64$ or $(2a + 1)^2 \leqslant 41$, so $a = -3, -2, -1, 0, 1, 2$. For $\mu = 1$, $\mu(2\omega - 1) - 4\lambda = a + b\omega$ with $b \equiv 2 \bmod 4$ so $| a + b\omega |^2 > 16$. The same is true for $\mu = 3$. For $\mu = 2$, $2(2\omega - 1) - 4\lambda = a + b\omega$ with $b \equiv 0 \bmod 4$. For $| a + b\omega |^2 \leqslant 16$ we need $\lambda = \omega + n$ with $n \in Z$ but all $(\omega + n)/2$ are singular and, in particular $(2, \omega + n) \neq \mathcal{O}$. Suppose now $\mu = \omega + n$, $n \in Z$. Then $\mu(2\omega - 1) - 4\lambda = (2n + 1)\,\omega - 2r - n - 4\lambda$, where $r = (m + 1)/4$. Let $\lambda = x + y\omega$, $x, y \in \mathbf{Z}$. Then $\mu(2\omega - 1) - 4\lambda = a + b\omega$ with $a = -2r - n - 4x$, $b = 2n + 1 - 4y$. If $| a + b\omega |^2 \leqslant 16$, we must have $b = \pm 1$, $-3 \leqslant a \leqslant 2$. Now

$$\overline{\mu(2\omega - 1) - 4\lambda} = -\bar{\mu}(2\omega - 1) - 4\bar{\lambda} \qquad \text{since} \quad 2\omega - 1 = \sqrt{-m}.$$

Therefore μ, λ satisfy the inequality if and only if $-\bar{\mu}, \bar{\lambda}$ do. If $\mu = \omega + n$, then $-\bar{\mu} = \omega - 1 - n$. Thus it is sufficient to do the case where n is even. In this case $y = n/2$, $b = 1$, and $a = 0, \pm 2$. Also $n = 0, \pm 2$. Now

$$| \omega |^2 = \frac{m + 1}{4}, \; | \omega + 2 |^2 = \frac{m + 25}{4}, \; | \omega - 2 |^2 = \frac{m + 9}{4}.$$

If either of n or a is $+2$, then

$$| \mu(2\omega - 1) - 4\lambda |^2 + | \mu |^2 \geqslant \frac{m + 1}{4} + \frac{m + 25}{4} = \frac{m + 13}{2}.$$

But $(m + 13)/2 \leqslant 16$ implies $m \leqslant 15$. Now $m \equiv -1 \bmod 8$, so $r = (m + 1)/4$ is even. Thus $a = -2r - n - 4x \equiv -n \bmod 4$. Therefore either $n = a = 0$ or $n = a = -2$. If $n = a = 0$, then $x = -r/2$, $y = 0$, so $\mu = \omega$, $\lambda = -r/2$. If $n = a = -2$, then $x = 1 - r/2$, $y = n/2 = -1$, so $\mu = \omega - 2$, $\lambda = 1 - (r/2) - \omega$. In these cases we have $(\mu, \lambda) \neq \mathcal{O}$. In general, if $\alpha \in \mathcal{O}$, $q \in \mathbf{Z}$, then $(\alpha, q) = \mathcal{O}$ if and only if $(N\alpha, q) = 1$. In one direction this is clear since $N\alpha = \bar{\alpha}\alpha \in (\alpha, q)$. If now there is some prime p dividing $N\alpha$ and q, then $(\alpha, q) \neq \mathcal{O}$ otherwise there would be some $\beta, \gamma \in \mathcal{O}$ with $\beta\alpha + \gamma q = 1$. Therefore $\bar{\alpha} = \beta N\alpha + \gamma\bar{\alpha}q$, so $p \mid \bar{\alpha}$. Taking conjugates, $p \mid \alpha$, so $(\alpha, q) \subset p\mathcal{O}$.

Now for $\mu = \omega$, $\lambda = -r/2$, $(\mu, \lambda) = (\omega, r/2)$, but $N\omega = r$, so $(N\omega, r/2) = r/2 \neq 1$ unless $r = 0, 2$ when $m = 1, 7$. For

$$\mu = \omega - 2, \qquad \lambda = 1 - \frac{r}{2} - \omega, \qquad (\mu, \lambda) = (\mu, \lambda + \mu) = \left(\omega - 2, \frac{r}{2} + 1\right).$$

Now $N(\omega - 2) = (9 + m)/4 = 2 + r$, so

$$\left(N(\omega - 2), \frac{r}{2} + 1\right) = \frac{r}{2} + 1 \neq 1 \qquad \text{for} \quad r \neq 0.$$

COROLLARY 9.11. *If $m \equiv -1 \bmod 8$ and $m \geqslant 23$, there is no edge on the line $\eta = \sqrt{m}/4$ and the only vertices on this line are the singular points $(\omega + n)/2$ for $n \in Z$. If an edge meets this line, it meets it at one of these singular points.*

The only values of m not covered by the above results are $m = 1, 2, 3, 7, 11, 15$. These will all be treated individually below.

10. The Case $Q(\sqrt{-2})$

In this case, as in the cases $m = 1, 3$ previously considered, B is defined by the $S_{\mu,\lambda}$ with $\mu = 1$. The region Q is shown in Fig. 1, the dotted edges being spurious ones. The only true vertex in Q is the upper right corner $v = (1 + \omega)/2$, with $\omega = \sqrt{-2}$. The corresponding point on $S_{1,0}$ has $\zeta = \frac{1}{2}$. To check these assertions, it is sufficient to check that the two edges do form the boundaries between $S_{1,0}$ and $S_{1,1}$, $S_{1,\omega}$, respectively, and then to determine all $S_{\mu,\lambda}$ covering the vertex $((1 + \omega)/2, \frac{1}{2})$. The first assertion is clear. For the second, we must consider the inequality $|\mu((1 + \omega)/2) - \lambda|^2 + \frac{1}{4}|\mu|^2 < 1$ or $|\mu(1 + \omega) - 2\lambda|^2 + |\mu|^2 < 4$. If $\mu = a + b\omega$, $|\mu|^2 = a^2 + 2b^2$, so $\mu = 1, \omega, \omega + 1, \omega - 1$. For $\mu = 1$, we get $|1 + \omega - 2\lambda|^2 < 3$ which is impossible. If $\mu = \omega$, $|\omega - 2 - 2\lambda|^2 < 2$ is again impossible.

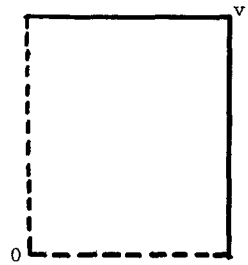

FIGURE 1

If $\mu = \omega + 1$, $| 1 + 2\omega - 2 - 2\lambda |^2 < 1$ and if $\mu = \omega - 1$, $| -3 - 2\lambda |^2 < 1$. These also are impossible.

The only 2-cell mod $\mathcal{O}$ is $S_{1,0}$ corresponding to the generator $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The vertical edge gives the relation $(TA)^3 = J$ of Corollary 9.3. We must now determine the relation corresponding to the horizontal edge. This has its apex at $(z, \zeta) = (\omega/2, 1/\sqrt{2})$. If this lies on $S_{\mu,\lambda}$, we have $| \mu(\omega/2) - \lambda |^2 + \frac{1}{2} | \mu |^2 \leqslant 1$ or $| \mu\omega - 2\lambda |^2 + 2 | \mu |^2 \leqslant 4$. Thus $\mu = 1$ or ω. If $\mu = 1$, $| \omega - 2\lambda |^2 \leqslant 2$, so $\lambda = 0$ or ω. If $\mu = \omega$, $| -2 - 2\lambda |^2 = 0$ or $\lambda = -1$. Therefore the $S_{\mu,\lambda}$ containing the horizontal edge have $\lambda/\mu = 0/1$, $-1/\omega$, $\omega/1$ in linear order. Choose $\tau_1 = A$, $\tau_2 = \begin{pmatrix} +1 & \omega \\ \omega & -1 \end{pmatrix}$, $\tau_3 = \begin{pmatrix} 0 & -1 \\ 1 & \omega \end{pmatrix} = AU$ where $U = \begin{pmatrix} 0 & \omega \\ 1 & 1 \end{pmatrix}$. The relation is $[A^{-1}][\tau_1\tau_2^{-1}][\tau_2\tau_3^{-1}][AU] = 1$. Now

$$\tau_1\tau_2^{-1} = \begin{pmatrix} \omega & 1 \\ -1 & -\omega \end{pmatrix} = JU^{-1}AU \qquad \text{and} \qquad \tau_2\tau_3^{-1} = \begin{pmatrix} 0 & +1 \\ -1 & \omega \end{pmatrix} = JAU^{-1},$$

so the relation is $A^{-1}JU^{-1}AU \cdot JAU^{-1} \cdot AU = 1$ or $(AU^{-1}AU)^2 = J$ using $A^2 = J$. The edges obtained by applying the symmetries π are all equivalent to the original one mod $\mathcal{O}$ so we have found all relations.

THEOREM 10.1. *Let $\mathcal{O}$ be the ring of integers of $Q(\sqrt{-2})$. Let $\omega = \sqrt{-2}$. Then $SL(2, \mathcal{O})$ is generated by the elements*

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with the relations

$$J^2 = 1, \quad J \text{ central}, \quad TU = UT, \quad A^2 = J,$$
$$(TA)^3 = J, \quad \text{and} \quad (AU^{-1}AU)^2 = J.$$

This follows immediately from Theorem 4.18.

COROLLARY 10.2. *If* $K = Q(\sqrt{-2})$, *then*

$$SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})] = \mathbf{Z} \oplus \mathbf{Z}/6\mathbf{Z}$$

generated by U *and* T.

To get a presentation of $GL(2, \mathcal{O})$, we add one new generator $E = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and new relations

$$ETE = T^{-1}, \qquad EUE = U^{-1}, \qquad EJ = JE, \qquad EAE = JA.$$

COROLLARY 10.3. *If* $K = Q(\sqrt{-2})$, *then*

$$GL(2, \mathcal{O})/[GL(2, \mathcal{O}), GL(2, \mathcal{O})] = (\mathbf{Z}/2\mathbf{Z})^3$$

generated by E, U, T.

To get $\widetilde{GL}(2, \mathcal{O})$ we add a new generator F with the relations $F^2 = 1$, $FEF = E$, $FTF = T$, $FUF = U^{-1}$, $FAF = A$, $FJF = J$.

11. THE CASE $Q(\sqrt{-5})$

In this case, Bianchi's calculations [2] show that we need three 2-cells modulo symmetries to define B. The corresponding values λ/μ are (I) $0/1$, (II) $\omega/2 = (0, \sqrt{5}/2)$, and (III) $(\omega - 4)/2\omega = (\frac{1}{2}, 2/\sqrt{5})$. The first two of these can be anticipated from Lemmas 9.2 and 9.4. The third could then easily be found by the methods of Section 8. The region Q is shown in Fig. 2, the dotted edges being spurious. The only true vertices in Q are the singular points s and the vertices p and v. The equations of the lines may be found by subtracting the equations of

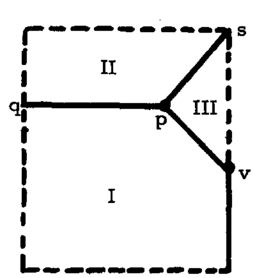

FIGURE 2

$S_{1,0}$, $S_{2,\omega}$, $S_{2\omega,\omega-4}$ as in Section 8. They are $qp: \sqrt{5}\eta = 2$, $ps: \eta = \sqrt{5}\xi$, and $pv: \sqrt{5}\xi + 4\eta = 2\sqrt{5}$. From this we find the coordinates (ξ, η) of the vertices to be

$$p = \left(\frac{2}{5}, \frac{2}{\sqrt{5}}\right) = \frac{2+2\omega}{5}, \quad v = \left(\frac{1}{2}, \frac{3\sqrt{5}}{8}\right) = \frac{4+3\omega}{8}, \quad s = \left(0, \frac{\sqrt{5}}{2}\right) = \frac{\omega}{2}.$$

The spurious vertex q is $(0, 2/\sqrt{5}) = 2\omega/5$. The point on ∂B over p has $\zeta = 1/5$ while that over v has $\zeta = \sqrt{3}/8$. We now must find all $S_{\mu,\lambda}$ covering or containing these vertices. The point $(z, \zeta) = ((2 + 2\omega)/5, \frac{1}{5})$ lying over p lies on $S_{\mu,\lambda}$ for

$$\frac{\lambda}{\mu} = \frac{0}{1}, \frac{\omega}{2}, \frac{2+2\omega}{5}, \frac{-2}{\omega}, \frac{2\omega}{\omega+4}, \frac{-5}{2\omega-2}, \frac{\omega-4}{2\omega}$$

while the point $(z, \zeta) = ((4 + 3\omega)/8, \sqrt{3}/8)$ lying over v lies on $S_{\mu,\lambda}$ for

$$\frac{\lambda}{\mu} = \frac{0}{1}, \frac{1}{1}, \frac{2\omega}{\omega+4}, \frac{-\omega-4}{\omega-4}, \frac{\omega-4}{2\omega}.$$

We see this by looking for solutions of the inequalities

$$|\mu(2+2\omega) - 5\lambda| + |\mu|^2 \leqslant 25 \quad \text{and} \quad |\mu(4+3\omega) - 8\lambda|^2 + 3|\mu|^2 \leqslant 64$$

and eliminating any with $(\mu, \lambda) \neq \mathcal{O}$. At the same time we check that no vertex can be covered, so we have found B. We will omit the details of the calculation since they are straightforward and quite tedious. We can now find the $S_{\mu,\lambda}$ containing the edge over pv by seeing which $S_{\mu,\lambda}$ contain both p and v. These are $\lambda/\mu = 0/1$, $2\omega/(\omega + 4)$, $(\omega - 4)/2\omega$. The horizontal edge has as vertices p and $-\bar{p}$, since the part lying in Q is half of the full edge. The vertex over $-\bar{p}$ lies on $S(\lambda/\mu)$ if and only if p lies on $S(-\bar{\lambda}/\bar{\mu})$. To find the $S_{\mu,\lambda}$ containing the horizontal edge, we look for those λ/μ with p on $S(\lambda/\mu)$ and on $S(-\bar{\lambda}/\bar{\mu})$. These are $0/1$, $-2/\omega$ and $\omega/2$. For the edge over ps we look for those λ/μ with the vertex over p on $S(\lambda/\mu)$ and with $|\mu((\omega + 1)/2) - \lambda|^2 = 1$, i.e., $|\mu(\omega + 1) - 2\lambda|^2 = 4$. We get $\lambda/\mu = \omega/2$, $-5/(2\omega - 2)$, $(\omega - 4)/2\omega$ in their correct order.

Now, by applying the symmetries we see that all 2-cells are represented mod $\mathcal{O}$ by $0/1$, $\pm\omega/2$, $(\pm\omega \pm 4)/2\omega$ or by $0/1$, $\omega/2$, $(\omega \pm 4)/2\omega$, these being distinct mod $\mathcal{O}$. As generators for $SL(2, \mathcal{O})$ we choose

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} -\omega & 2 \\ 2 & \omega \end{pmatrix}, \quad C = \begin{pmatrix} -\omega-4 & -2\omega \\ 2\omega & \omega-4 \end{pmatrix}, \quad D = \begin{pmatrix} 4-\omega & -2\omega \\ 2\omega & \omega+4 \end{pmatrix}.$$

Note that $A^2 = B^2 = CD = J$. The vertical edges yield the relation $(TA)^3 = J$ by Corollary 9.3. For the horizontal edge in Q, choose $\tau_1 = A$, $\tau_2 = \left(\begin{smallmatrix} 2 & \omega \\ \omega & -2 \end{smallmatrix}\right)$, $\tau_3 = B$. Then $\tau_1\tau_2^{-1} = JB$ and $\tau_2\tau_3^{-1} = JA$, so the relation $[\tau_1^{-1}][\tau_1\tau_2^{-1}][\tau_2\tau_3^{-1}][\tau_3] = 1$ is $JA \cdot JB \cdot JA \cdot B = 1$ or $(AB)^2 = J$. For the edge over pv, choose $\tau_1 = A$, $\tau_2 = \left(\begin{smallmatrix} 2\omega & \omega-4 \\ \omega+4 & 2\omega \end{smallmatrix}\right)$, $\tau_3 = C$ getting $JA \cdot TDT^{-1} \cdot JA \cdot C = 1$ or $ATDT^{-1}AC = 1$ or $ACA = JTCT^{-1}$. For the edge over ps, choose $\tau_1 = B$, $\tau_2 = \left(\begin{smallmatrix} -5 & -2\omega-2 \\ 2\omega-2 & 5 \end{smallmatrix}\right)$, $\tau_3 = C$ getting $JB \cdot TDT^{-1} \cdot JUBU^{-1} \cdot C = 1$ or $BTDT^{-1}UBU^{-1}C = 1$ or $UBU^{-1}CB = JTCT^{-1}$.

To find the remaining edge relations, we apply the symmetries in the 4 group π to those already obtained. If

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{let} \qquad \bar{X} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$$

$$X' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} X \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = J\begin{pmatrix} -a & -d \\ c & -d \end{pmatrix}.$$

Then $\bar{J} = J' = J$, $\bar{T} = T$, $T' = \bar{T}' = T^{-1}$, $\bar{U} = U' = U^{-1}$, $\bar{U}' = U$, $\bar{A} = A$, $A' = \bar{A}' = JA$, $\bar{B} = UBU^{-1}$, $B' = JUBU^{-1}$, $\bar{B}' = JB$, $\bar{C} = JD$, $C' = JTDT^{-1}$, $\bar{C}' = TCT^{-1}$, $\bar{D} = JC$, $D = JTCT^{-1}$, $\bar{D}' = TDT^{-1}$.

The horizontal edge is invariant under $z \to -\bar{z}$, so it is sufficient to look at its transform by $z \to \bar{z}$. The relation $(AB)^2 = J$ becomes $(\bar{A}\bar{B})^2 = \bar{J}$ or $(AUBU^{-1})^2 = J$. The relation $ACA = JTCT^{-1}$ gives $\bar{A}\bar{C}\bar{A} = \bar{J}\bar{T}\bar{C}\bar{T}^{-1}$ or $AJDA = JTJDT^{-1}$ which is equivalent to the inverse of the original relation. Thus the only new relation we could get from this is

$$A'C'A' = J'T'C'T'^{-1} \qquad \text{or} \qquad JA \cdot JTDT^{-1} \cdot JA = JT^{-1} \cdot JTDT^{-1} \cdot T$$

or

$$JD = ATDT^{-1}A$$

but this is again equivalent to the original relation. The relation $UBU^{-1}CB = JTCT^{-1}$ gives $\overline{UBU^{-1}CB} = \overline{JTCT^{-1}}$ or $BJDUBU^{-1} = JTJDT^{-1}$ which again is the inverse of the original relation. It only remains to try

$$U'B'U'^{-1}C'B' = J'T'C'T'^{-1}$$

or

$$U^{-1} \cdot JUBU^{-1} \cdot U \cdot JTDT^{-1} \cdot JUBU^{-1} = JT^{-1} \cdot JTDT^{-1} \cdot T$$

or

$$BTDT^{-1}JUBU^{-1} = D$$

which is again equivalent to the original relation.

THEOREM 11.1. *Let $\mathcal{O}$ be the ring of integers of $Q(\sqrt{-5})$. Let $\omega = \sqrt{-5}$. Then $SL(2, \mathcal{O})$ is generated by the elements*

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} -\omega & 2 \\ 2 & \omega \end{pmatrix}, \quad C = \begin{pmatrix} -\omega - 4 & -2\omega \\ 2\omega & \omega - 4 \end{pmatrix}$$

with relations

$$J^2 = 1, \quad J \text{ central}, \quad TU = UT, \quad A^2 = J, \quad B^2 = J,$$

$$(TA)^3 = J, \quad (AB)^2 = J, \quad (AUBU^{-1})^2 = J, \quad ACA = JTCT^{-1},$$

$$UBU^{-1}CB = JTCT^{-1}.$$

From this we easily deduce the result quoted in [1]. The elementary subgroup $E(2, \mathcal{O})$ of $SL(2, \mathcal{O})$ is the subgroup generated by all $\left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right)$ for $s \in \mathcal{O}$. Clearly $J, T, U, A \in E(2, \mathcal{O})$. Since $\left(\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right) = A\left(\begin{smallmatrix} 1 & -s \\ 0 & 1 \end{smallmatrix}\right) A^{-1}$, we see that $E(2, \mathcal{O})$ is the subgroup generated by J, T, U, A.

COROLLARY 11.2. *If $\mathcal{O}$ is the ring of integers of $Q(\sqrt{-5})$, then $SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})] = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/6\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ generated by U, C, T, B, and $SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})]\, E(2, \mathcal{O}) = \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ generated by C and B.*

Also, $[SL(2, \mathcal{O}), SL(2, \mathcal{O})]\, E(2, \mathcal{O})$ is the smallest normal subgroup containing $E(2, \mathcal{O})$.

To get a presentation for $GL(2, \mathcal{O})$, we add one new generator $E = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and new relations $E^2 = 1$, $ETE = T^{-1}$, $EUE = U^{-1}$, $EJ = JE$, $EAE = JA$, $EBE = JUBU^{-1}$, and $ECE = TC^{-1}T^{-1}$.

COROLLARY 11.3. *If $K = Q(\sqrt{-5})$, then*

$$GL(2, \mathcal{O})/[GL(2, \mathcal{O}), GL(2, \mathcal{O})] = (\mathbf{Z}/2\mathbf{Z})^5$$

generated by E, U, T, B, C.

To get $\widetilde{GL}(2, \mathcal{O})$, we add a new generator F with the relations $F^2 = 1$, $FEF = E, FTF = T, FUF = U^{-1}, FAF = A, FJF = J, FBF = UBU^{-1}$, $FCF = C^{-1}$.

12. The Case $Q(\sqrt{-6})$

Bianchi's results [2] show that in this case the 2-cells are given by the following values of λ/μ: (I) $0/1$, (II) $(1 + \omega)/2 = (\frac{1}{2}, \sqrt{6}/2)$, and (III) $-5/2\omega = (0, 5/2\sqrt{6})$. The region Q is shown in Fig. 3. The lines are given by

$$pq\colon \eta = 2\sqrt{6}/5, \qquad ps\colon \sqrt{6}\xi + \eta = \sqrt{6}/2, \qquad \text{and} \qquad pw\colon \xi + \sqrt{6}\eta = 5/2.$$

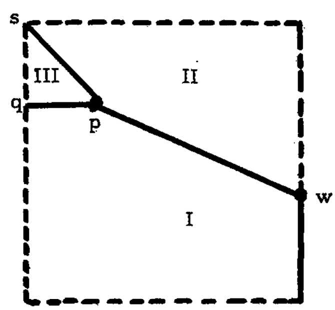

Figure 3

The vertices are

$$s = \omega/2 = (0, \sqrt{6}/2), \qquad p = (1 + 4\omega)/10 = (1/10, 2\sqrt{6}/5),$$

and

$$w = (3 + 2\omega)/6 = (1/2, 2/\sqrt{6}).$$

The spurious vertex $q = 2\omega/5 = (0, 2\sqrt{6}/5)$. The vertex s is singular. The point on ∂B over p has $\zeta = \sqrt{3}/10$ while that over w has $\zeta = \sqrt{3}/6$. The $S(\lambda/\mu)$ containing the edge over ps are given by the following values of λ/μ (in order): $(\omega + 1)/2$, $(\omega - 5)/(2\omega + 2)$, $-5/2\omega$. For pq we get $0/1$, $2\omega/5$, $-5/2\omega$ while for pw we get $0/1$, $-2/(\omega - 1)$, $(1 + \omega)/3$, $-3/(\omega - 1)$, $(1 + \omega)/2$. The vertical edge through w has already been treated in Section 9.

THEOREM 12.1. *Let $\mathcal{O}$ be the ring of integers of $Q(\sqrt{-6})$. Let $\omega = \sqrt{-6}$. Then $SL(2, \mathcal{O})$ is generated by the elements*

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} -1-\omega & 2-\omega \\ 2 & 1+\omega \end{pmatrix}, \quad C = \begin{pmatrix} 5 & -2\omega \\ 2\omega & 5 \end{pmatrix},$$

with the relations

$$J^2 = 1, \quad J \text{ central}, \quad TU = UT, \quad A^2 = J, \quad B^2 = J,$$
$$(TA)^3 = J, \quad AC = CA, \quad T^{-1}CTUBU^{-1} = BC, \quad (ATB)^3 = J,$$
$$(ATUBU^{-1})^3 = J.$$

The last relation is obtained by applying the symmetries. The other relations so obtained are all consequences of the given ones.

COROLLARY 12.2. *If $\mathcal{O}$ is the ring of integers of $Q(\sqrt{-6})$, then $SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})] = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/6\mathbf{Z}$ generated by C, U, T while $SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})]\, E(2, \mathcal{O}) = \mathbf{Z}$ generated by C.*

Again $[SL(2, \mathcal{O}), SL(2, \mathcal{O})]\, E(2, \mathcal{O})$ is the smallest normal subgroup containing $E(2, \mathcal{O})$.

To get a presentation of $GL(2, \mathcal{O})$ we add the new generator $E = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the new relations $E^2 = 1$, $ETE = T^{-1}$, $EUE = U^{-1}$, $EJ = JE$, $EAE = JA$, $EBE = JTUBU^{-1}T^{-1}$, and $ECE = C^{-1}$.

COROLLARY 12.3. *If $K = Q(\sqrt{-6})$, then*

$$GL(2, \mathcal{O})/[GL(2, \mathcal{O}), GL(2, \mathcal{O})] = (\mathbf{Z}/2\mathbf{Z})^4$$

generated by E, U, T, C.

To get $\widetilde{GL}(2, \mathcal{O})$ we add a new generator F with the relations $F^2 = 1$, $FEF = E, FTF = T, FUF = U^{-1}, FAF = A, FJF = J, FBF = UBU^{-1}$, $FCF = C^{-1}$.

13. THE CASE $Q(\sqrt{-7})$

In this case, all the 2-cells have $\mu = 1$ [2]. The region Q is shown in Fig. 4, the 2-cells appearing there being given by $\lambda/\mu = 0/1$ and $\omega/1$, where $\omega = \frac{1}{2}(1 + \sqrt{-7})$. The vertex v is given by $v = (3\omega + 2)/7 =$

$(1/2, 3/2\sqrt{7})$. The spurious vertex q is $\omega/2 = (1/4, \sqrt{7}/4)$. The edge qv is $\xi + \sqrt{7}\eta = 2$. The point on ∂B over v has $\zeta = \sqrt{3/7}$. The $S(\lambda/\mu)$ containing the edge over qv are given by the following values of λ/μ (in order): $0/1$, $-1/(\omega - 1)$, $\omega/1$.

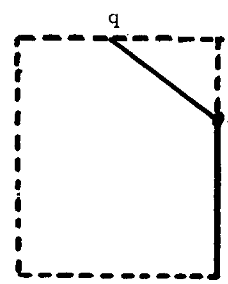

FIGURE 4

THEOREM 13.1. *Let $\mathcal{O}$ be the ring of integers of $Q(\sqrt{-7})$. Let $\omega = \frac{1}{2}(1 + \sqrt{-7})$. Then $SL(2, \mathcal{O})$ is generated by the elements*

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with the relations

$$J^2 = 1, \quad J \text{ central}, \quad TU = UT, \quad A^2 = J, \quad (AT)^3 = J, \quad (ATU^{-1}AU)^2 = J.$$

COROLLARY 13.2. *If $K = Q(\sqrt{-7})$, then*

$$SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})] = \mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z}$$

generated by U and T.

To get a presentation of $GL(2, \mathcal{O})$ we add the new generator $E = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the new relations $E^2 = 1$, $ETE = T^{-1}$, $EUE = U^{-1}$, $EJ = JE$, $EAE = JA$.

COROLLARY 13.3. *If $K = Q(\sqrt{-7})$, then*

$$GL(2, \mathcal{O})/[GL(2, \mathcal{O}), GL(2, \mathcal{O})] = (\mathbf{Z}/2\mathbf{Z})^3$$

generated by E, U, T.

To get $\widetilde{GL}(2, \mathcal{O})$, we add a new generator F with the relations $F^2 = 1$, $FEF = E$, $FTF = T$, $FUF = TU^{-1}$, $FAF = A$, $FJF = J$.

14. The Case $Q(\sqrt{-11})$

Again, all the 2-cells have $\mu = 1$ [2]. The region Q is again given by Fig. 4, the 2-cells appearing there being given by $\lambda/\mu = 0/1$ and $\omega/1$, where $\omega = \frac{1}{2}(1 + \sqrt{-11})$. The vertex v is given by $v = (5\omega + 3)/11 = (1/2, 5/2\sqrt{11})$. The spurious vertex q is $\omega/2 = (1/4, \sqrt{11}/4)$. The edge qv is $\xi + \sqrt{11}\eta = 3$. The point on ∂B over v has $\zeta = \sqrt{2/11}$. The $S(\lambda/\mu)$ containing the edge over qv are given by the following values of λ/μ (in order): $0/1$, $-1/(\omega - 1)$, $\omega/2$, $-2/(\omega - 1)$, $\omega/1$.

Theorem 14.1. *Let $\mathcal{O}$ be the ring of integers of $Q(\sqrt{-11})$. Let $\omega = \frac{1}{2}(1 + \sqrt{-11})$. Then $SL(2, \mathcal{O})$ is generated by the elements*

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with the relations

$$J^2 = 1, \quad J \text{ central}, \quad TU = UT, \quad A^2 = J, \quad (AT)^3 = J, \quad (ATU^{-1}AU)^3 = J.$$

The only difference between this case and the case $Q(\sqrt{-7})$ is the exponent in the last relation.

Corollary 14.2. *If $K = Q(\sqrt{-11})$, then*

$$SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})] = \mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$$

generated by U and T.

To get a presentation of $GL(2, \mathcal{O})$, we add the new generator $E = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the new relations $E^2 = 1$, $ETE = T^{-1}$, $EUE = U^{-1}$, $EJ = JE$, and $EAE = JA$.

Corollary 14.3. *If $K = Q(\sqrt{-11})$, then*

$$GL(2, \mathcal{O})/[GL(2, \mathcal{O}), GL(2, \mathcal{O})] = (\mathbf{Z}/2\mathbf{Z})^2$$

generated by E, U.

To get $\widetilde{GL}(2, \mathcal{O})$ we add a new generator F with the relations $F^2 = 1$, $FEF = E$, $FTF = T$, $FUF = TU^{-1}$, $FAF = A$, $FJF = J$.

15. The Case $Q(\sqrt{-15})$

In this case, the 2-cells are given by the following values of λ/μ: (I) $0/1$, (II) $(\omega - 4)/(2\omega - 1)$, where $\omega = \frac{1}{2}(1 + \sqrt{-15})$. We also need $-4/(2\omega - 1)$ but this is obtained from (II) by symmetry and translation. The region Q is shown in Fig. 5. The singular vertex s is $s = \omega/2 =$

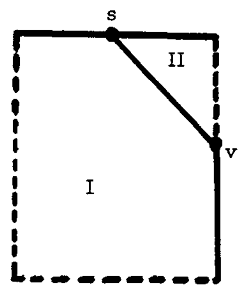

Figure 5

$(1/4, \sqrt{15}/4)$. The vertex v is $v = (3\omega + 2)/7 = (1/2, 3\sqrt{15}/14)$. The point of ∂B over v has $\zeta = \sqrt{3}/7$. The edge sv is given by $\xi + 7\eta/\sqrt{15} = 2$. The $S(\lambda/\mu)$ containing the edge over sv are given by $\lambda/\mu = 0/1$, $(2\omega - 1)/(\omega + 3)$, $(\omega - 4)/(2\omega - 1)$. The $S(\lambda/\mu)$ for the edge joining s to $s - 1$ are given by $\lambda/\mu = 0/1, (2\omega - 1)/4, -4/(2\omega - 1)$.

Theorem 15.1. *Let $\mathcal{O}$ be the ring of integers of $Q(\sqrt{-15})$. Let $\omega = \frac{1}{2}(1 + \sqrt{-15})$. Then $SL(2, \mathcal{O})$ is generated by the elements*

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 4 & 1 - 2\omega \\ 2\omega - 1 & 4 \end{pmatrix}$$

with the relations

$$J^2 = 1, \quad J \text{ central}, \quad TU = UT, \quad A^2 = J, \quad (TA)^3 = J, \quad AC = CA$$
$$UCUAT = TAUCU.$$

Corollary 15.2. *If $K = Q(\sqrt{-15})$, then*

$$SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})] = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/12\mathbf{Z}$$

generated by C, U, T while $SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})]\, E(2, \mathcal{O}) = \mathbf{Z}$ generated by C.

Once again $[SL(2, \mathcal{O}), SL(2, \mathcal{O})]\, E(2, \mathcal{O})$ is the smallest normal subgroup containing $E(2, \mathcal{O})$.

To get a presentation of $GL(2, \mathcal{O})$ we add the new generator $E = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$ and the new relations $E^2 = 1$, $ETE = T^{-1}$, $EUE = U^{-1}$, $EJ = JE$, $EAE = JA$, and $ECE = C^{-1}$.

COROLLARY 15.3. *If $K = Q(\sqrt{-15})$, then*

$$GL(2, \mathcal{O})/[GL(2, \mathcal{O}), GL(2, \mathcal{O})] = (\mathbf{Z}/2\mathbf{Z})^4$$

generated by E, U, T, C.

To get $\widetilde{GL}(2, \mathcal{O})$, we add a new generator F with the relations $F^2 = 1$, $FEF = E$, $FTF = T$, $FUF = TU^{-1}$, $FAF = A$, $FJF = J$, $FCF = C^{-1}$.

16. THE CASE $Q(\sqrt{-19})$

The 2-cells are given by the following values of λ/μ: (I) $0/1$, (II) $\omega/2$, where $\omega = \frac{1}{2}(1 + \sqrt{-19})$. The region Q is shown in Fig. 6. The vertices

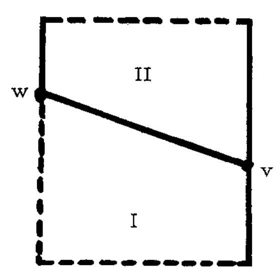

FIGURE 6

are $v = (\omega - 4)/(2\omega - 1) = (1/2, 7/2\sqrt{19})$ and $w = -4/(2\omega - 1) = (0, 4/\sqrt{19})$. The edge vw is $\xi + \sqrt{19}\eta = 4$. The point on ∂B over w has $\zeta = \sqrt{3/19}$ while that over v has $\zeta = \sqrt{2/19}$. The $S(\lambda/\mu)$ containing the edge over vw are given by $\lambda/\mu = 0/1$, $-2/(\omega - 1)$, $\omega/2$. The relations corresponding to the other edges are given in Section 9.

THEOREM 16.1. *Let $\mathcal{O}$ be the ring of integers of $Q(\sqrt{-19})$. Let $\omega = \frac{1}{2}(1 + \sqrt{-19})$. Then $SL(2, \mathcal{O})$ is generated by the elements*

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 1-\omega & 2 \\ 2 & \omega \end{pmatrix}$$

with the relations

$$J^2 = 1, \quad J \text{ central}, \quad TU = UT, \quad A^2 = J, \quad (TA)^3 = J,$$
$$B^3 = J, \quad (BT^{-1})^3 = 1, \quad (AB)^2 = J, \quad (AT^{-1}UBU^{-1})^2 = J.$$

COROLLARY 16.2. *If $K = Q(\sqrt{-19})$, then*

$$SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})] = \mathbf{Z}$$

generated by U.

Also the smallest normal subgroup of $SL(2, \mathcal{O})$ containing $E(2, \mathcal{O})$ is $SL(2, \mathcal{O}) = [SL(2, \mathcal{O}), SL(2, \mathcal{O})]\, E(2, \mathcal{O})$.

To get a presentation of $GL(2, \mathcal{O})$, we add the new generator $E = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the new relations $E^2 = 1$, $ETE = T^{-1}$, $EUE = U^{-1}$, $EJ = JE$, $EAE = JA$, and $EBE = JT^{-1}UBU^{-1}$.

COROLLARY 16.3. *If $K = Q(\sqrt{-19})$, then*

$$GL(2, \mathcal{O})/[GL(2, \mathcal{O}), GL(2, \mathcal{O})] = (\mathbf{Z}/2\mathbf{Z})^2$$

generated by E and U.

To get $\widetilde{GL}(2, \mathcal{O})$, we add a new generator F with the relations $F^2 = 1$, $FEF = E$, $FTF = T$, $FUF = TU^{-1}$, $FAF = A$, $FJF = J$, $FBF = JUB^{-1}U^{-1}T$.

17. FINAL REMARKS

In all cases examined above, the following results hold:

(1) $SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})]$ has rank equal to the class number of K,

(2) The element U represents an element of infinite order in this group,

(3) $SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O}) E(2, \mathcal{O})$ has rank $h - 1$,

(4) $[SL(2, \mathcal{O}), SL(2, \mathcal{O})] E(2, \mathcal{O})$ is the smallest normal subgroup of $SL(2, \mathcal{O})$ containing $E(2, \mathcal{O})$,

(5) $GL(2, \mathcal{O})/[GL(2, \mathcal{O}), GL(2, \mathcal{O})]$ is finite,

(6) $GL(2, \mathcal{O})/[GL(2, \mathcal{O}), GL(2, \mathcal{O})]$ is an elementary Abelian 2-group for $K \neq Q(\sqrt{-1}), Q(\sqrt{-3})$.

It would be tempting to conjecture these results hold for all K. However, recent calculations of Mennicke for $Q(\sqrt{-10})$ show that (1), (3), (5), and (6) are false in general. Serre [15] has shown that the rank of $SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})]$ is at least h, the class number of K. His method is topological and is based on the observation that $H/SL(2, \mathcal{O})$ is a rational homology manifold which can be compactified by h tori. He also constructs an explicit subgroup of rank h and thus shows that (1) implies (5). Presumably one could also verify (2) by his methods but I have not checked this.

It is easy to see that if (2) holds, the rank of $SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})] E(2, \mathcal{O})$ is one less than that of $SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})]$. The group $E(2, \mathcal{O})$ is generated by

$$J, T, U, A \qquad \text{for} \quad K \neq Q(\sqrt{-1}), Q(\sqrt{-3}).$$

These satisfy the relations $TU = UT$, J central, $J^2 = 1$, $A^2 = J$, and $(TA)^3 = J$. Cohn [6] has shown that these relations give a presentation of $E(2, \mathcal{O})$ except when K is Euclidean, i.e., when $K = Q(\sqrt{-m})$, $m = 1, 2, 3, 7, 11$. Therefore $E(2, \mathcal{O})/[E(2, \mathcal{O}), E(2, \mathcal{O})]$ is a quotient of $\mathbf{Z} \oplus \mathbf{Z}/12\mathbf{Z}$ and is equal to this group except in the 5 Euclidean cases. Therefore the image of $E(2, \mathcal{O})$ in $SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})]$ has rank $\leqslant 1$ and this rank is 1 if and only if (2) holds.

Serre [15] has observed that the "extended" groups of Bianchi [2] turn out to be groups generated by reflections ("Coxeter groups") for small values of m. This gives an alternative method for constructing presentations of these groups. However, Mennicke has shown that this assertion is false for large values of m.

It is not hard to see why (6) holds for small values of m. The relations for $E(2, \mathcal{O})$ show that $J \in [GL(2, \mathcal{O}), GL(2, \mathcal{O})]$. Therefore the Abelianization of $GL(2, \mathcal{O})$ is the same as that of $PGL(2, \mathcal{O}) = GL(2, \mathcal{O})/\{1, J\}$. In the cases considered here with $K \neq Q(\sqrt{-1}), Q(\sqrt{-3})$, $PGL(2, \mathcal{O})$ is generated by involutions. In fact, let S be the set of all $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathcal{O})$ satisfying either $a = \pm d$ or $b = \pm c$. Given

$X \in S$, we can assume $a = \pm d$ by using AX if necessary. Similarly we can assume $a = -d$ using EX if $a = d$. If $a = -d$, tr $X = 0$, but any 2×2 matrix satisfies the equation $X^2 - \text{tr}(X)\, X + \det X = 0$. Thus $X \in GL(2, \mathcal{O})$, and tr $X = 0$ imply $X^2 = 1$ or J. This shows that the subgroup of $PGL(2, \mathcal{O})$ generated by S is also generated by involutions. All generators in the above calculations lie in S but there is no reason why this should continue to hold for large values of m.

The calculations obviously have not been pushed far enough to make any reasonable conjecture about the rank of $SL(2, \mathcal{O})/[SL(2, \mathcal{O}), SL(2, \mathcal{O})]$. The length of the calculation increases so rapidly with the discriminant that machine computation seems to be the only reasonable approach.

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